Appendix

A Mathematical Treatment of the Baumol-Tobin and Tobin Mean-Variance Models

Baumol-Tobin Model of Transactions Demand for Money

The basic idea behind the Baumol-Tobin model was laid out in the chapter. Here we explore the mathematics that underlie the model. The assumptions of the model are as follows:

1. An individual receives income of T_0 at the beginning of every period.
2. An individual spends this income at a constant rate, so at the end of the period, all income T_0 has been spent.
3. There are only two assets—cash and bonds. Cash earns a nominal return of zero, and bonds earn an interest rate *i*.
4. Every time an individual buys or sells bonds to raise cash, a fixed brokerage fee of *b* is incurred.

Let us denote the amount of cash that the individual raises for each purchase or sale of bonds as *C*, and *n* = the number of times the individual conducts a transaction in bonds. As we saw in Figure 21-3 in the chapter, where T_0 = 1000, *C* = 500, and *n* = 2,

$$n = \frac{T_0}{C}$$

Because the brokerage cost of each bond transaction is *b*, the total brokerage costs for a period are

$$nb = \frac{bT_0}{C}$$

Not only are there brokerage costs, but there is also an opportunity cost to holding cash rather than bonds. This opportunity cost is the bond interest rate *i* times average cash balances held during the period, which, from the discussion in the chapter, we know is equal to *C*/2. The opportunity cost is then

$$\frac{iC}{2}$$

Combining these two costs, we have the total costs for an individual equal to

$$\text{Costs} = \frac{bT_0}{C} + \frac{iC}{2}$$

The individual wants to minimize costs by choosing the appropriate level of *C*. This is accomplished by taking the derivative of costs with respect to *C* and setting it to zero.[1] That is,

$$\frac{d\ \text{Costs}}{dc} = \frac{-bT_0}{C^2} + \frac{i}{2} = 0$$

Solving for *C* yields the optimal level of *C*:

$$C = \sqrt{\frac{2bT_0}{i}}$$

Because money demand M^d is the average desired holding of cash balances $C/2$,

$$M^d = \frac{1}{2}\sqrt{\frac{2bT_0}{i}} = \sqrt{\frac{bT_0}{2i}} \qquad (1)$$

This is the famous *square root rule*.[2] It has these implications for the demand for money:

1. The transactions demand for money is negatively related to the interest rate *i*.
2. The transactions demand for money is positively related to income, but there are economies of scale in money holdings—that is, the demand for money rises less than proportionally with income. For example, if T_0 quadruples in Equation 1, the demand for money only doubles.
3. A lowering of the brokerage costs due to technological improvements would decrease the demand for money.
4. There is no money illusion in the demand for money. If the price level doubles, T_0 and *b* will double. Equation 1 then indicates that *M* will double as well. Thus the demand for real money balances remains unchanged, which makes sense because neither the interest rate nor real income has changed.

Tobin Mean-Variance Model

Tobin's mean-variance analysis of money demand is just an application of the basic ideas in the theory of portfolio choice. Tobin assumes that the utility that people derive from their assets is positively related to the expected return on their portfolio of assets and is negatively related to the riskiness of this portfolio as represented by the variance (or standard deviation) of its returns. This framework implies that an individual has indifference curves that can be drawn as in Figure 21A-1. Notice that these indifference curves slope upward because an individual is willing to accept more risk if offered a higher expected return. In addition, as we go to higher indifference curves, utility is higher because for the same level of risk, the expected return is higher.

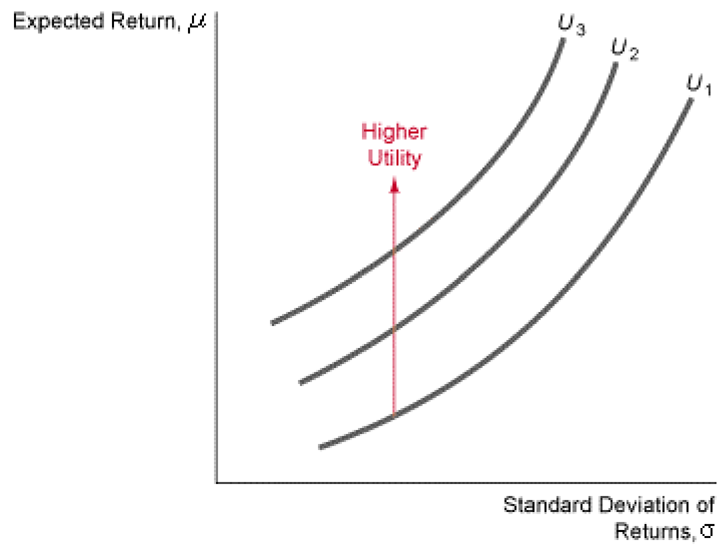

Figure 21A-1 *Indifference Curves in a Mean-Variance Model*
The indifference curves are upward-sloping, and higher indifference curves indicate that utility is higher. In other words, $U_3 > U_2 > U_1$.

Tobin looks at the choice of holding money, which earns a certain zero return, or bonds, whose return

$$R_B = i + g$$

where i = interest rate on the bond and g = capital gain. Tobin also assumes that the expected capital gain is zero[3] and its variance is σ_g^2.

That is,

$$E(g) = 0 \qquad \text{and so} \qquad E(R_B) = i + 0 = i$$

$$Var(g) = E[g - E(g)]^2 = E(g^2) = \sigma_g^2$$

where E = expectation of the variable inside the parentheses and Var = variance of the variable inside the parentheses.

If A is the fraction of the portfolio put into bonds ($0 \le A \le 1$) and $1 - A$ is the fraction of the portfolio held as money, the return R on the portfolio can be written as

$$R = AR_B + (1 - A)(0) = AR_B = A(i + g)$$

Then the mean and variance of the return on the portfolio, denoted respectively as μ and σ^2, can be calculated as follows:

$$\mu = E(R) = E(AR_B) = AE(R_B) = Ai$$

$$\sigma^2 = E(R - \mu)^2 = E[A(i + g) - Ai]^2 = E(Ag)^2 = A^2E(g^2) = A^2\sigma_g^2$$

Taking the square root of both sides of the equation directly above and solving for A yields

$$A = \frac{i}{\sigma_g}\sigma \qquad (2)$$

Substituting for A in the equation $\mu = Ai$ using the preceding equation gives us

$$\mu = \frac{i}{\sigma_g}\sigma \qquad (3)$$

Equation 3 is known as the *opportunity locus* because it tells us the combinations of μ and σ that are feasible for the individual. This equation is written in a form in which the μ variable corresponds to the y axis and the σ variable to the x axis. The opportunity locus is a straight line going through the origin with a slope of i/σ_g. It is drawn in the top half of Figure 21A-2 along with the indifference curves from Figure 21A-1.

The highest indifference curve is reached at point B, the tangency of the indifference curve and the opportunity locus. This point determines the optimal level of risk σ^* in the figure. As Equation 2 indicates, the optimal level of A, A^* is

$$A^* = \frac{\sigma^*}{\sigma_g}$$

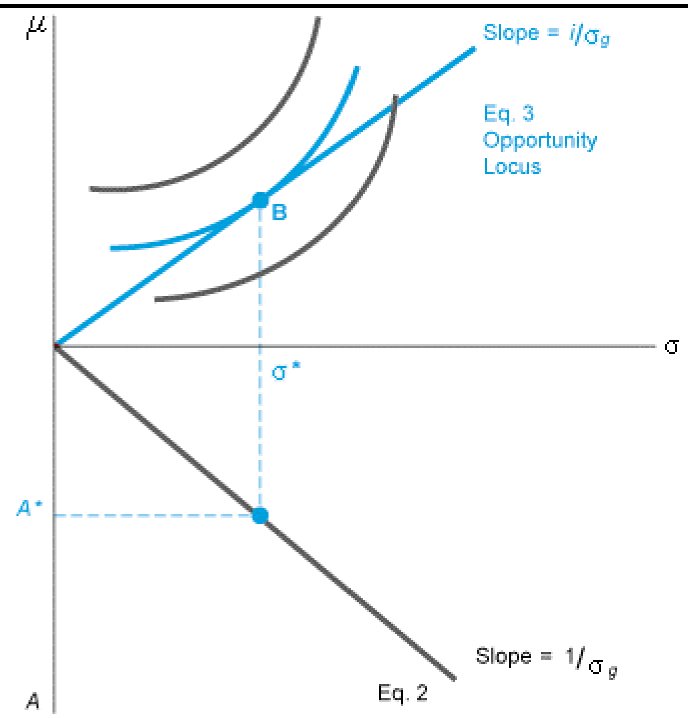

Figure 21A-2 *Optimal Choice of the Fraction of the Portfolio in Bonds*
The highest indifference curve is reached at a point B, the tangency of the indifference curve with the opportunity locus. This point determines the optimal risk σ^*, and using Equation 2 in the bottom half of the figure, we solve for the optimal fraction of the portfolio in bonds A^*.

This equation is solved in the bottom half of Figure 21A-2. Equation 2 for A is a straight line through the origin with a slope of $1/\sigma_g$. Given σ^*, the value of A read off this line is the optimal value A^*. Notice that the bottom part of the figure is drawn so that as we move down, A is increasing.

Now let's ask ourselves what happens when the interest rate increases from i_1 to i_2. This situation is shown in Figure 21A-3. Because σ_g is unchanged, the Equation 2 line in the bottom half of the figure does not change. However, the slope of the opportunity locus does increase as i increases. Thus the opportunity locus rotates up and we move to point C at the tangency of the new opportunity locus and the indifference curve. As you can see, the optimal level of risk increases from σ_1^* to σ_2^* and the optimal fraction of the portfolio in bonds rises from A_1^* to A_2^*. The result is that as the interest rate on bonds rises, the demand for money falls; that is, $1 - A$, the fraction of the portfolio held as money, declines.[4]

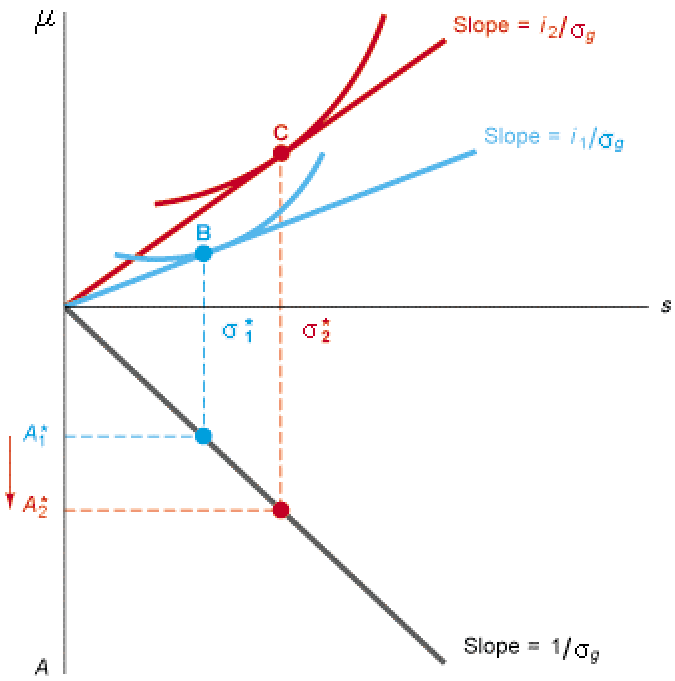

Figure 21A-3 *Optimal Choice of the Fraction of the Portfolio in Bonds as the Interest Rate Rises*
The interest rate on bonds rises from i_1 to i_2, rotating the opportunity locus upward. The highest indifference curve is now at point C where it is tangent to the new opportunity locus. The optimal level of risk rises from σ_1^* to σ_2^* and then Equation 2, in the bottom half of the figure, shows that the optimal fraction of the portfolio in bonds rises from A_1^* to A_2^*.

Tobin's model then yields the same result as Keynes's analysis of the speculative demand for money: It is negatively related to the level of interest rates. This model, however, makes two important points that Keynes's model does not:

1. Individuals diversify their portfolios and hold money *and* bonds at the same time.
2. Even if the expected return on bonds is greater than the expected return on money, individuals will still hold money as a store of wealth because its return is more certain.

NOTES

[1] To minimize costs, the second derivative must be greater than zero. We find that it is, because

$$\frac{d^2 \text{ Costs}}{dC^2} = \frac{-2}{C^3}(-bT_0) = \frac{2bT_0}{C^3} > 0$$

[2] An alternative way to get Equation 1 is to have the individual maximize profits, which equal the interest on bonds minus the brokerage costs. The average holding of bonds over a period is just

$$\frac{T_0}{2} - \frac{C}{2}$$

Thus profits are

$$\text{Profits} = -\frac{i}{2}(T_0 - C) - \frac{bT_0}{C}$$

Then

$$\frac{d \text{ Profits}}{dC} = \frac{-i}{2} + \frac{bT_0}{C^2} = 0$$

This equation yields the same square root rule as Equation 1.

[3] This assumption is not critical to the results. If $E(g) \neq 0$, it can be added to the interest term i, and the analysis proceeds as indicated.

[4] The indifference curves have been drawn so that the usual result is obtained that as i goes up, A^* goes up as well. However, there is a subtle issue of income versus substitution effects. If, as people get wealthier, they are willing to bear less risk, and if this income effect is larger than the substitution effect, then it is possible to get the opposite result that as i increases, A^* declines. This set of conditions is unlikely, which is why the figure is drawn so that the usual result is obtained. For a discussion of income versus substitution effects, see David Laidler, *The Demand for Money: Theories and Evidence*, 4th ed. (New York: HarperCollins, 1993).