CHAPTER 14

SECTIONS 14.11 and 14.12

14.11 Flux and Circulation

In many branches of engineering and physics we encounter the concepts of flux and circulation. In this section we discuss the relationships between flux and divergence and between circulation and curl. We find a physical setting mentioned in Section 14.1 of the textbook most useful in developing these ideas.

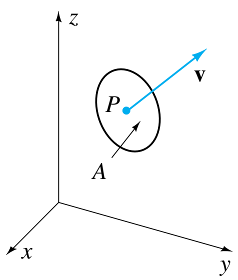


FIGURE 14.51 Vector $\mathbf{v}$ represents direction of gas flow at point P. A is a unit area at P perpendicular to $\mathbf{v}$

Fluid Flow

Consider a gas flowing through a region D of space. At time t and point $P(x, y, z)$ in D, gas flows through P with velocity $\mathbf{v}(x, y, z, t)$. If A is a unit area around P perpendicular to $\mathbf{v}$ (Figure 14.51) and $\rho(x, y, z, t)$ is density of the gas at P, then the amount of gas crossing A per unit time is $\rho\mathbf{v}$. At every point P in D, then, the vector $\rho\mathbf{v}$ is such that its direction $\mathbf{v}$ gives velocity of gas flow, and its magnitude $\rho|\mathbf{v}|$ describes the mass of gas flowing in that direction per unit time.

Consider some surface S in D (Figure 14.52). If $\hat{\mathbf{n}}$ is a unit normal to S, then $\rho\mathbf{v} \cdot \hat{\mathbf{n}}$ is the component of $\rho\mathbf{v}$ normal to the surface S. If dS is an element of area on S, then $\rho\mathbf{v} \cdot \hat{\mathbf{n}}\, dS$ describes the mass of gas flowing through dS per unit time. Consequently,

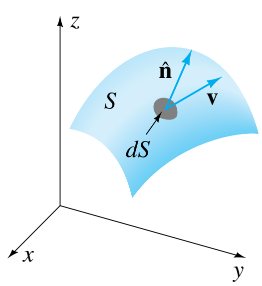


FIGURE 14.52 Flux for a surface S is the mass of gas passing through S per unit time

$$\iint_S \rho\mathbf{v} \cdot \hat{\mathbf{n}}\, dS$$

is the mass of gas flowing through S per unit time. This quantity is called **flux** for surface S.

If S is a closed surface (Figure 14.53) and $\hat{\mathbf{n}}$ is the unit outer normal to S, then

$$\oiint_S \rho\mathbf{v} \cdot \hat{\mathbf{n}}\, dS$$

is the mass of gas flowing out of surface S per unit time. If this flux (for closed S) is positive, then there is a net outward flow of gas through S (i.e., more gas is leaving the volume bounded by S than is entering); if the flux is negative, the net flow is inward.

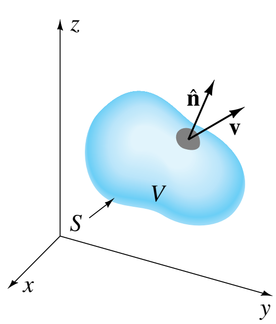


FIGURE 14.53 Flux for a closed surface is the mass of gas flowing out of the surface per unit time

If we apply the divergence theorem to the flux integral over the closed surface S, we have

$$\oiint_S \rho\mathbf{v} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot (\rho\mathbf{v})\, dV. \qquad (14.60)$$

Now the flux (on the left side of this equation) is the mass of gas per unit time leaving S. In order for the right side to represent the same quantity, $\nabla \cdot (\rho\mathbf{v})$ must be interpreted as the mass of gas leaving unit volume per unit time, because then $\nabla \cdot (\rho\mathbf{v})\, dV$ represents the mass per unit time leaving dV, and the triple integral is the mass per unit time leaving V.

We have obtained, therefore, an interpretation of the divergence of $\rho\mathbf{v}$. The divergence of $\rho\mathbf{v}$ is the flux per unit volume per unit time at a point: the mass of gas leaving unit volume in unit time. We can use this idea of flux to derive the *equation of continuity* for fluid flow. The triple integral

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = \frac{\partial}{\partial t} \iiint_V \rho\, dV$$

measures the time rate of change of mass in a volume V. If this triple integral is positive, then there is a net inward flow of mass; if it is negative, the net flow is outward. We conclude, therefore, that this triple integral must be the negative of the flux for the volume V; that is,

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla \cdot (\rho \mathbf{v})\, dV$$

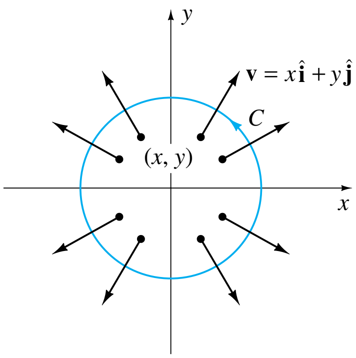
or

$$\iiint_V \left[\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t}\right] dV = 0.$$

If $\nabla \cdot (\rho \mathbf{v})$ and $\partial \rho / \partial t$ are continuous functions, then this equation can hold for arbitrary volume V only if

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0. \qquad (14.61)$$

FIGURE 14.54 Circulation for radial gas flow is zero

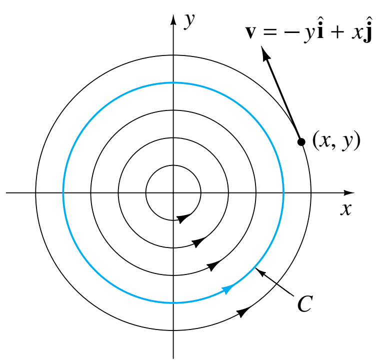


This equation, called the **equation of continuity**, expresses conservation of mass; it is basic to all fluid flow.

If C is a closed curve in the flow region D, then the **circulation** of the flow for the curve C is defined by

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{r}. \qquad (14.62)$$

FIGURE 14.55 Circulation for circular gas flow is not zero

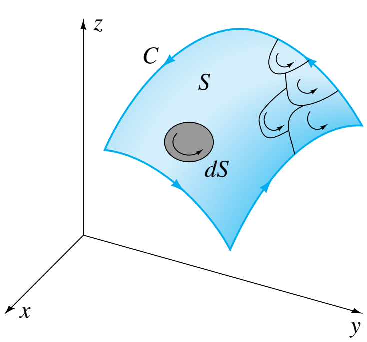


To obtain an intuitive feeling for Γ, we consider two very simple two-dimensional flows. First, suppose $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, so that all particles of gas flow along radial lines directed away from the origin (Figure 14.54). In this case, the line integral defining Γ is independent of path and $\Gamma = 0$ for any curve whatsoever.

Second, suppose $\mathbf{v} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, so that all particles of the gas flow counterclockwise around circles centred at the origin (Figure 14.55). In this case Γ does not generally vanish. In particular, if C is the circle $x^2 + y^2 = r^2$, then $\mathbf{v}$ and $d\mathbf{r}$ are parallel, and

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{r} = \oint_C |\mathbf{v}|\, ds = \oint_C \sqrt{y^2 + x^2}\, ds = \oint_C r\, ds = 2\pi r^2.$$

FIGURE 14.56 $\nabla \times \mathbf{v}$ is an indication of the circulatory nature of a flow



These two flow patterns indicate perhaps that circulation is a measure of the tendency for the flow to be circulatory. If we apply Stokes's theorem to the circulation integral for the closed curve C, we have

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, dS, \qquad (14.63)$$

where S is any surface in the flow with boundary C. If the right side of this equation is to represent the circulation for C also, then $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, dS$ must be interpreted as the circulation for the curve bounding dS (or simply for dS itself). Then the addition process of the surface integral (Figure 14.56) gives the circulation around C, the circulation around all internal boundaries cancelling. But if $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}\, dS$ is the circulation for dS, then it follows that $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}$ must be the circulation for unit area perpendicular to $\hat{\mathbf{n}}$. Thus $\nabla \times \mathbf{v}$ describes the circulatory nature of the flow $\mathbf{v}$; its component $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}$ in any direction describes the circulation for unit area perpendicular to $\hat{\mathbf{n}}$.

Electromagnetic Theory

The concepts of flux and circulation also play a prominent role in electromagnetic theory. For example, suppose a dielectric contains a charge distribution of density ρ (charge per unit volume). This charge produces an electric field represented by the electric displacement vector **D**. If S is a surface in the dielectric, then the flux of **D** through S is defined as

$$\iint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, dS,$$

and in the particular case in which S is closed, as

$$\oiint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, dS.$$

Gauss's law states that this flux integral must be equal to the total charge enclosed by S. If V is the region enclosed by S, we can write

$$\oiint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \rho\, dV. \tag{14.64}$$

On the other hand, if we apply the divergence theorem to the flux integral, we have

$$\oiint_S \mathbf{D} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{D}\, dV.$$

Consequently,

$$\iiint_V \nabla \cdot \mathbf{D}\, dV = \iiint_V \rho\, dV \quad \Longrightarrow \quad \iiint_V (\nabla \cdot \mathbf{D} - \rho)\, dV = 0.$$

If $\nabla \cdot \mathbf{D}$ and ρ are continuous functions, then the only way this equation can hold for arbitrary volume V in the dielectric is if

$$\nabla \cdot \mathbf{D} = \rho. \tag{14.65}$$

This is the first of Maxwell's equations for electromagnetic fields.

Another of Maxwell's equations can be obtained using Stokes's theorem. The flux through a surface S of a magnetic field **B** is defined by

$$\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS.$$

If **B** is a changing field, then an induced electric field intensity **E** is created. Faraday's induction law states that the time rate of change of the flux of **B** through S must be equal to the negative of the line integral of **E** around the boundary C of S:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS. \tag{14.66}$$

But Stokes's theorem applied to the line integral also gives

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, dS.$$

It follows, therefore, that if S is stationary,

$$\iint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, dS = \iint_S -\frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}}\, dS \quad \Longrightarrow \quad \iint_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot \hat{\mathbf{n}}\, dS = 0.$$

Once again, this equation can hold for arbitrary surfaces S, if $\nabla \times \mathbf{E}$ and $\partial \mathbf{B}/\partial t$ are continuous, only if

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{14.67}$$

14.12 Vector Analysis in Orthogonal Coordinates

We used polar, cylindrical, and spherical coordinates in Chapter 13 to evaluate multiple integrals; they are useful in many other contexts. In this section we discuss the gradient, divergence, and curl in these coordinate systems. We do so in detail for spherical coordinates, and develop similar results for cylindrical coordinates and other orthogonal coordinate systems in the exercises.

Spherical coordinates $(\Re, \phi, \theta)$ are related to Cartesian coordinates by the equations

$$x = \Re \sin\phi \cos\theta, \quad y = \Re \sin\phi \sin\theta, \quad z = \Re \cos\phi. \tag{14.68}$$

With the restrictions, $\Re \geq 0$, $0 \leq \phi \leq \pi$, $-\pi < \theta \leq \pi$, every point in space (except those on the z-axis) has exactly one set of spherical coordinates. Inverse to transformation 14.68 are the equations

$$\Re = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \mathrm{Cos}^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) \pm \pi, \tag{14.69}$$

where in the formula for θ, it must be decided, depending on x, y, and z, whether π should be added, π should be subtracted, or neither should be done.

To express a scalar function $f(x, y, z)$ in terms of spherical coordinates, we use equations 14.68 to write $f(\Re \sin\phi \cos\theta, \Re \sin\phi \sin\theta, \Re \cos\phi)$. Expressing vectors in spherical coordinates is more complicated. Vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ form a basis for all vectors in space; that is, every vector $\mathbf{v}$ can be expressed as a linear combination $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, where v_x, v_y, and v_z are the Cartesian components of $\mathbf{v}$. To find the corresponding basis for spherical coordinates, we begin by calling a line with equations y = constant, z = constant a coordinate curve in Cartesian coordinates; it is a straight line parallel to the x-axis. Every point on the line has the same y- and z-coordinates; only the x-coordinate varies. A unit tangent vector to this line in the direction of increasing x is $\hat{\mathbf{i}}$. Similarly, $\hat{\mathbf{j}}$ is tangent to the coordinate curve (line) x = constant, z = constant, and $\hat{\mathbf{k}}$ is tangent to x = constant, y = constant, and this is true at every point in space.

FIGURE 14.57a Coordinate curve along which only $\Re$ varies

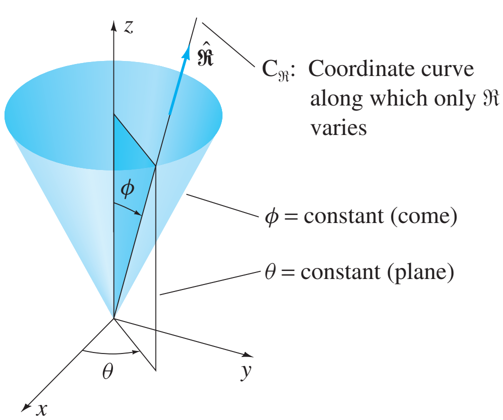


FIGURE 14.57b Coordinate curve along which only ϕ varies

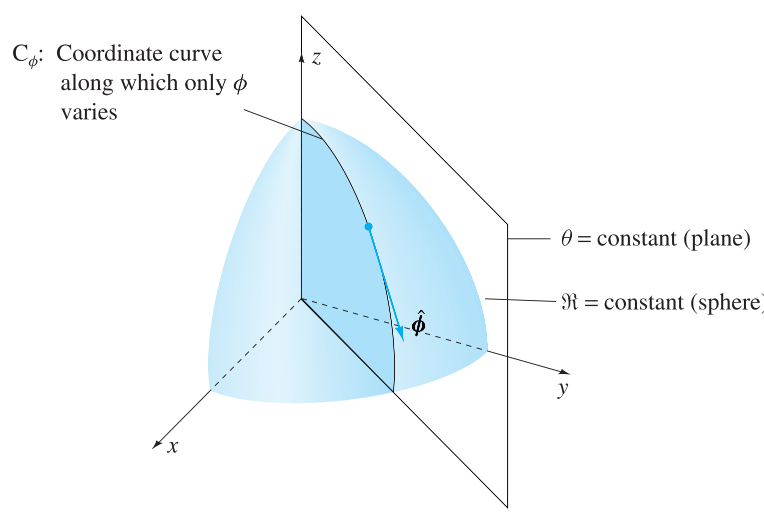


FIGURE 14.57c Coordinate curve along which only θ varies

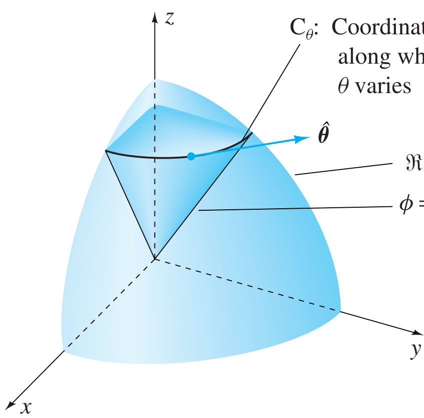
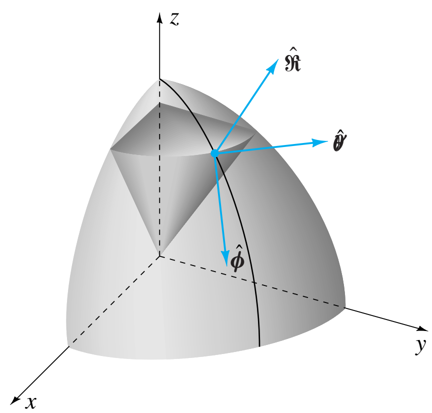


FIGURE 14.58 Unit tangent vectors to coordinate curves are mutually perpendicular



The equations ϕ = constant, θ = constant define a (half) straight line $C_\Re$ from the origin; it is called a *coordinate curve in spherical coordinates* along which only $\Re$ varies (Figure 14.57a). Parametric equations for $C_\Re$ are equations 14.68 with ϕ and θ fixed. If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is the position vector of points on $C_\Re$, then a unit tangent vector to $C_\Re$, pointing in the direction in which $\Re$ increases, at any point on $C_\Re$ is

$$\hat{\Re} = \frac{\partial\mathbf{r}/\partial\Re}{|\partial\mathbf{r}/\partial\Re|} = \frac{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)}{\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi}}$$
$$= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi). \quad (14.70a)$$

A unit tangent vector to the coordinate curve C_ϕ in Figure 14.57b along which ϕ varies, but $\Re$ and θ are constant, and points in the direction in which ϕ increases, is

$$\hat{\boldsymbol{\phi}} = \frac{\partial\mathbf{r}/\partial\phi}{|\partial\mathbf{r}/\partial\phi|} = \frac{(\Re\cos\phi\cos\theta, \Re\cos\phi\sin\theta, -\Re\sin\phi)}{\sqrt{\Re^2\cos^2\phi\cos^2\theta + \Re^2\cos^2\phi\sin^2\theta + \Re^2\sin^2\phi}}$$
$$= (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi). \quad (14.70b)$$

Finally, a unit tangent vector to the coordinate curve C_θ in Figure 14.57c is

$$\hat{\boldsymbol{\theta}} = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = \frac{(-\Re\sin\phi\sin\theta, \Re\sin\phi\cos\theta, 0)}{\sqrt{\Re^2\sin^2\phi\sin^2\theta + \Re^2\sin^2\phi\cos^2\theta}}$$
$$= (-\sin\theta, \cos\theta, 0). \quad (14.70c)$$

All three unit vectors $\hat{\Re}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$ are drawn at the same point in Figure 14.58. They are mutually perpendicular, as can also be seen algebraically by noting that

$$\hat{\Re}\cdot\hat{\boldsymbol{\theta}} = \hat{\Re}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = 0. \quad (14.71)$$

Because this is true at every point in space (except on the z-axis), spherical coordinates are said to constitute a set of **orthogonal curvilinear coordinates**. It is also a right-handed coordinate system in that $\hat{\Re}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}$. Unlike Cartesian coordinates where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ always have the same direction, directions of $\hat{\Re}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$ vary from point to point.

Vectors $\hat{\Re}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$ form a basis for all vectors in space; that is, every vector $\mathbf{v}$ can be expressed in the form $\mathbf{v} = v_\Re\hat{\Re} + v_\phi\hat{\boldsymbol{\phi}} + v_\theta\hat{\boldsymbol{\theta}}$, where $v_\Re$, v_ϕ, and v_θ are called the *spherical* *components* of $\mathbf{v}$. Given the Cartesian components of $\mathbf{v}$, we can find its spherical components at any point $(\Re, \phi, \theta)$ by taking scalar products with $\hat{\boldsymbol{\Re}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$:

$$v_\Re = \mathbf{v} \cdot \hat{\boldsymbol{\Re}} = (v_x, v_y, v_z) \cdot (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$= v_x \sin\phi\cos\theta + v_y \sin\phi\sin\theta + v_z\cos\phi, \tag{14.72a}$$

$$v_\phi = \mathbf{v} \cdot \hat{\boldsymbol{\phi}} = (v_x, v_y, v_z) \cdot (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

$$= v_x \cos\phi\cos\theta + v_y \cos\phi\sin\theta - v_z\sin\phi, \tag{14.72b}$$

$$v_\theta = \mathbf{v} \cdot \hat{\boldsymbol{\theta}} = (v_x, v_y, v_z) \cdot (-\sin\theta, \cos\theta, 0)$$

$$= -v_x\sin\theta + v_y\cos\theta. \tag{14.72c}$$

These equations define the spherical components of a vector in terms of its Cartesian components. It is understood that v_x, v_y, and v_z are to be expressed in terms of $\Re$, ϕ, and θ. We can invert these equations and express v_x, v_y, and v_z in terms of $v_\Re$, v_ϕ, and v_θ. Scalar products of $\mathbf{v} = v_\Re\hat{\boldsymbol{\Re}} + v_\phi\hat{\boldsymbol{\phi}} + v_\theta\hat{\boldsymbol{\theta}}$ with $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ give

$$v_x = v_\Re\hat{\boldsymbol{\Re}} \cdot \hat{\mathbf{i}} + v_\phi\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{i}} + v_\theta\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{i}}$$

$$= v_\Re \sin\phi\cos\theta + v_\phi\cos\phi\cos\theta - v_\theta\sin\theta, \tag{14.73a}$$

$$v_y = v_\Re\hat{\boldsymbol{\Re}} \cdot \hat{\mathbf{j}} + v_\phi\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{j}} + v_\theta\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{j}}$$

$$= v_\Re \sin\phi\sin\theta + v_\phi\cos\phi\sin\theta + v_\theta\cos\theta, \tag{14.73b}$$

$$v_z = v_\Re\hat{\boldsymbol{\Re}} \cdot \hat{\mathbf{k}} + v_\phi\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{k}} + v_\theta\hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{k}}$$

$$= v_\Re\cos\phi - v_\phi\sin\phi. \tag{14.73c}$$

There is an important difference between Cartesian and spherical components of a vector. No matter where the tail of a vector is placed, its Cartesian components are always the same; spherical components 14.72 of a vector depend on where the tail is placed. This is a direct result of the fact that $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ always have the same direction, whereas $\hat{\boldsymbol{\Re}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$ do not. This is illustrated in the following example.

EXAMPLE 14.29

What are the spherical components of the vector $\hat{\mathbf{i}}$? Evaluate these components at the points with Cartesian coordinates $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 1)$. In each case draw $\hat{\mathbf{i}}$, $\hat{\boldsymbol{\Re}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$ at the point.

SOLUTION Since Cartesian components of $\hat{\mathbf{i}}$ are $(1, 0, 0)$, equations 14.72 give

$$i_\Re = \sin\phi\cos\theta, \quad i_\phi = \cos\phi\cos\theta, \quad i_\theta = -\sin\theta.$$

Thus, $\hat{\mathbf{i}} = \sin\phi\cos\theta\,\hat{\boldsymbol{\Re}} + \cos\phi\cos\theta\,\hat{\boldsymbol{\phi}} - \sin\theta\,\hat{\boldsymbol{\theta}}$. Spherical coordinates of the point with Cartesian coordinates $(1, 0, 0)$ are $(1, \pi/2, 0)$ so that at this point, $\hat{\mathbf{i}} = \hat{\boldsymbol{\Re}}$ (Figure 14.59a). Spherical coordinates of the point with Cartesian coordinates $(0, 1, 0)$ are $(1, \pi/2, \pi/2)$, and at this point $\hat{\mathbf{i}} = -\hat{\boldsymbol{\theta}}$ (Figure 14.59b). Finally, the point with Cartesian coordinates $(1, 1, 1)$ has spherical coordinates $(\sqrt{3}, \mathrm{Cos}^{-1}(1/\sqrt{3}), \pi/4)$, and at this point

$$i_\Re = \sin[\mathrm{Cos}^{-1}(1/\sqrt{3})]\cos(\pi/4) = 1/\sqrt{3}, \quad i_\phi = (1/\sqrt{3})(1/\sqrt{2}) = 1/\sqrt{6}, \quad i_\theta = -1/\sqrt{2};$$

that is, $\hat{\mathbf{i}} = (1/\sqrt{6})(\sqrt{2}\hat{\boldsymbol{\Re}} + \hat{\boldsymbol{\phi}} - \sqrt{3}\hat{\boldsymbol{\theta}})$ (Figure 14.59c).

FIGURE 14.59a $\hat{\mathbf{i}} = \hat{\Re}$ at $(1, 0, 0)$

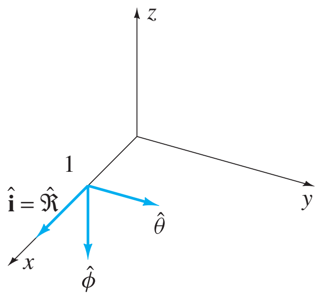


FIGURE 14.59b $\hat{\mathbf{i}} = -\hat{\boldsymbol{\theta}}$ at $(0, 1, 0)$

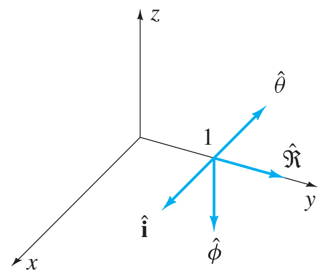
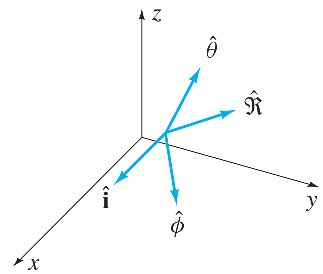


FIGURE 14.59c $\hat{\mathbf{i}} = \frac{1}{\sqrt{6}}(\sqrt{2}\hat{\Re} + \hat{\boldsymbol{\phi}} - \sqrt{3}\hat{\boldsymbol{\theta}})$ at $(1, 1, 1)$



Because spherical coordinates are orthogonal (equation 14.71), calculating the scalar product of two vectors with spherical components $\mathbf{u} = (u_\Re, u_\phi, u_\theta)$ and $\mathbf{v} = (v_\Re, v_\phi, v_\theta)$ is much like that with Cartesian components: multiply corresponding components and add,

$$\mathbf{u} \cdot \mathbf{v} = (u_\Re \hat{\Re} + u_\phi \hat{\boldsymbol{\phi}} + u_\theta \hat{\boldsymbol{\theta}}) \cdot (v_\Re \hat{\Re} + v_\phi \hat{\boldsymbol{\phi}} + v_\theta \hat{\boldsymbol{\theta}}) = u_\Re v_\Re + u_\phi v_\phi + u_\theta v_\theta. \tag{14.74}$$

Furthermore, using the facts that $\hat{\Re} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = \hat{\Re}$, and $\hat{\boldsymbol{\theta}} \times \hat{\Re} = \hat{\boldsymbol{\phi}}$, we obtain

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_\Re \hat{\Re} + u_\phi \hat{\boldsymbol{\phi}} + u_\theta \hat{\boldsymbol{\theta}}) \times (v_\Re \hat{\Re} + v_\phi \hat{\boldsymbol{\phi}} + v_\theta \hat{\boldsymbol{\theta}}) \\ &= (u_\phi v_\theta - u_\theta v_\phi)\hat{\Re} + (u_\theta v_\Re - u_\Re v_\theta)\hat{\boldsymbol{\phi}} + (u_\Re v_\phi - u_\phi v_\Re)\hat{\boldsymbol{\theta}}.\end{aligned}$$

This is more easily remembered in determinant form,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\Re} & \hat{\boldsymbol{\phi}} & \hat{\boldsymbol{\theta}} \\ u_\Re & u_\phi & u_\theta \\ v_\Re & v_\phi & v_\theta \end{vmatrix}. \tag{14.75}$$

Once again, the simplicity here is due to equations 14.71: spherical coordinates are orthogonal.

EXAMPLE 14.30

Find the scalar and vector products of the vectors $\mathbf{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - z^2\hat{\mathbf{k}}$ and $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ in spherical coordinates by (a) calculating the products in Cartesian coordinates and transforming to spherical coordinates and (b) transforming **u** and **v** to spherical components and using 14.74 and 14.75.

SOLUTION

(a) In Cartesian coordinates $\mathbf{u} \cdot \mathbf{v} = x^2 + y^2 - z^4$, and when we transform to spherical coordinates, $\mathbf{u} \cdot \mathbf{v} = \Re^2 \sin^2\phi - \Re^4 \cos^4\phi$. Furthermore,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & -z^2 \\ x & y & z^2 \end{vmatrix} = 2yz^2\hat{\mathbf{i}} - 2xz^2\hat{\mathbf{j}}.$$

Using 14.72 we can express this in terms of $\hat{\Re}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} = {} & (2yz^2 \sin\phi\cos\theta - 2xz^2 \sin\phi\sin\theta)\hat{\Re} + (2yz^2\cos\phi\cos\theta - 2xz^2\cos\phi\sin\theta)\hat{\boldsymbol{\phi}} \\ & + (-2yz^2\sin\theta - 2xz^2\cos\theta)\hat{\boldsymbol{\theta}}\end{aligned}$$

$$= 2\Re^2\cos^2\phi\sin\phi(\Re\sin\phi\sin\theta\cos\theta - \Re\sin\phi\cos\theta\sin\theta)\hat{\boldsymbol{\Re}}$$
$$+ 2\Re^2\cos^2\phi\cos\phi(\Re\sin\phi\sin\theta\cos\theta - \Re\sin\phi\cos\theta\sin\theta)\hat{\boldsymbol{\phi}}$$
$$- 2\Re^2\cos^2\phi(\Re\sin\phi\sin^2\theta + \Re\sin\phi\cos^2\theta)\hat{\boldsymbol{\theta}}$$
$$= -2\Re^3\sin\phi\cos^2\phi\,\hat{\boldsymbol{\theta}}.$$

(b) Spherical components of **u** and **v** are

$$\mathbf{u} = (x\sin\phi\cos\theta + y\sin\phi\sin\theta - z^2\cos\phi)\hat{\boldsymbol{\Re}} + (x\cos\phi\cos\theta + y\cos\phi\sin\theta + z^2\sin\phi)\hat{\boldsymbol{\phi}}$$
$$+ (-x\sin\theta + y\cos\theta)\hat{\boldsymbol{\theta}}$$
$$= (\Re\sin^2\phi\cos^2\theta + \Re\sin^2\phi\sin^2\theta - \Re^2\cos^3\phi)\hat{\boldsymbol{\Re}}$$
$$+ (\Re\sin\phi\cos\phi\cos^2\theta + \Re\sin\phi\cos\phi\sin^2\theta + \Re^2\sin\phi\cos^2\phi)\hat{\boldsymbol{\phi}}$$
$$+ (-\Re\sin\phi\sin\theta\cos\theta + \Re\sin\phi\sin\theta\cos\theta)\hat{\boldsymbol{\theta}}$$
$$= (\Re\sin^2\phi - \Re^2\cos^3\phi)\hat{\boldsymbol{\Re}} + (\Re\sin\phi\cos\phi + \Re^2\sin\phi\cos^2\phi)\hat{\boldsymbol{\phi}},$$

and similarly,

$$\mathbf{v} = (\Re\sin^2\phi + \Re^2\cos^3\phi)\hat{\boldsymbol{\Re}} + (\Re\sin\phi\cos\phi - \Re^2\sin\phi\cos^2\phi)\hat{\boldsymbol{\phi}}.$$

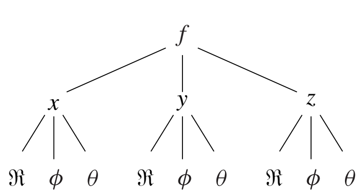
Consequently,

$$\mathbf{u}\cdot\mathbf{v} = (\Re\sin^2\phi - \Re^2\cos^3\phi)(\Re\sin^2\phi + \Re^2\cos^3\phi)$$
$$+ (\Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi)(-\Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi)$$
$$= \Re^2\sin^4\phi - \Re^4\cos^6\phi - \Re^4\sin^2\phi\cos^4\phi + \Re^2\sin^2\phi\cos^2\phi$$
$$= \Re^2\sin^2\phi(1 - \cos^2\phi) - \Re^4\cos^4\phi(1 - \sin^2\phi) - \Re^4\sin^2\phi\cos^4\phi + \Re^2\sin^2\phi\cos^2\phi$$
$$= \Re^2\sin^2\phi - \Re^4\cos^2\phi$$

and

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\boldsymbol{\Re}} & \hat{\boldsymbol{\phi}} & \hat{\boldsymbol{\theta}} \\ \Re\sin^2\phi - \Re^2\cos^3\phi & \Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi & 0 \\ \Re\sin^2\phi + \Re^2\cos^3\phi & -\Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi & 0 \end{vmatrix}$$
$$= [(\Re\sin^2\phi - \Re^2\cos^3\phi)(-\Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi)$$
$$- (\Re\sin^2\phi + \Re^2\cos^3\phi)(\Re^2\sin\phi\cos^2\phi + \Re\sin\phi\cos\phi)]\hat{\boldsymbol{\theta}}$$
$$= -2\Re^3\sin\phi\cos^2\phi\,\hat{\boldsymbol{\theta}}.$$

We now discuss the gradient, divergence, and curl in spherical coordinates. According to equations 14.72, the spherical components of the gradient vector $\nabla f = (\partial f/\partial x)\hat{\mathbf{i}} + (\partial f/\partial y)\hat{\mathbf{j}} + (\partial f/\partial z)\hat{\mathbf{k}}$ of a function $f(x, y, z)$ are



$$(\nabla f)_\Re = \frac{\partial f}{\partial x}\sin\phi\cos\theta + \frac{\partial f}{\partial y}\sin\phi\sin\theta + \frac{\partial f}{\partial z}\cos\phi, \qquad (14.76a)$$

$$(\nabla f)_\phi = \frac{\partial f}{\partial x}\cos\phi\cos\theta + \frac{\partial f}{\partial y}\cos\phi\sin\theta - \frac{\partial f}{\partial z}\sin\phi, \qquad (14.76b)$$

$$(\nabla f)_\theta = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta. \qquad (14.76c)$$

But with the schematic to the left

$$\frac{\partial f}{\partial \Re} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \Re} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \Re} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \Re} = \frac{\partial f}{\partial x}\sin\phi\cos\theta + \frac{\partial f}{\partial y}\sin\phi\sin\theta + \frac{\partial f}{\partial z}\cos\theta,$$

and similarly,

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\Re\cos\phi\cos\theta + \frac{\partial f}{\partial y}\Re\cos\phi\sin\theta - \frac{\partial f}{\partial z}\Re\sin\phi,$$

$$\frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x}\Re\sin\phi\sin\theta + \frac{\partial f}{\partial y}\Re\sin\phi\cos\theta.$$

Therefore,

$$\text{grad } f = \frac{\partial f}{\partial \Re}\hat{\boldsymbol{\Re}} + \frac{1}{\Re}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{\Re\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}. \tag{14.77}$$

Hence to find spherical components of the gradient of a function $f(x, y, z)$, we can transform its Cartesian components according to 14.76, or we can transform $f(x, y, z)$ to $f(\Re\cos\theta\sin\phi, \Re\sin\theta\sin\phi, \Re\cos\phi)$ and use 14.77.

EXAMPLE 14.31

Use equations 14.76 and 14.77 to find the spherical components of ∇f when $f(x, y, z) = x^2 + y^2 + z^3$.

SOLUTION With $\nabla f = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 3z^2\hat{\mathbf{k}}$, equations 14.76 give

$$\begin{aligned}
(\nabla f)_\Re &= 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta + 3z^2\cos\phi \\
&= 2\Re\sin^2\phi\cos^2\theta + 2\Re\sin^2\phi\sin^2\theta + 3\Re^2\cos^2\phi\cos\theta \\
&= 2\Re\sin^2\phi + 3\Re^2\cos^2\phi\cos\theta, \\
(\nabla f)_\phi &= 2x\cos\phi\cos\theta + 2y\cos\phi\sin\theta - 3z^2\sin\phi \\
&= 2\Re\sin\phi\cos\phi\cos^2\theta + 2\Re\sin\phi\cos\phi\sin^2\theta - 3\Re^2\sin\phi\cos^2\phi \\
&= 2\Re\sin\phi\cos\phi - 3\Re^2\sin\phi\cos^2\phi, \\
(\nabla f)_\theta &= -2x\sin\theta + 2y\cos\theta \\
&= -2\Re\sin\phi\sin\theta\cos\theta + 2\Re\sin\phi\sin\theta\cos\theta \\
&= 0.
\end{aligned}$$

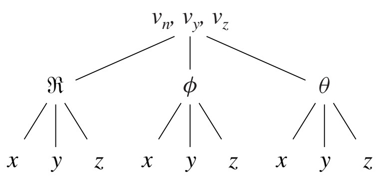
Thus, grad $f = (2\Re\sin^2\phi + 3\Re^2\cos^3\phi)\hat{\boldsymbol{\Re}} + (2\Re\sin\phi\cos\phi - 3\Re^2\sin\phi\cos^2\phi)\hat{\boldsymbol{\phi}}$. Alternatively, with $f(\Re\cos\theta\sin\phi, \Re\sin\theta\sin\phi, \Re\cos\phi) = \Re^2\sin^2\phi + \Re^3\cos^3\phi$, equations 14.77 give

$$\text{grad } f = (2\Re\sin^2\phi + 3\Re^2\cos^3\phi)\hat{\boldsymbol{\Re}} + \frac{1}{\Re}(2\Re^2\sin\phi\cos\phi - 3\Re^3\cos^2\phi\sin\phi)\hat{\boldsymbol{\phi}}.$$

The divergence of a vector function $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ is

$$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{14.78}$$

This can be changed to spherical coordinates by replacing x, y, and z according to 14.68. Alternatively, using the schematic to the left yields



$$\frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial \Re}\frac{\partial \Re}{\partial x} + \frac{\partial v_x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial v_x}{\partial \phi}\frac{\partial \phi}{\partial x}.$$

With 14.69,

$$\frac{\partial \Re}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{\Re \sin\phi\cos\theta}{\Re} = \sin\phi\cos\theta,$$

$$\frac{\partial \phi}{\partial x} = \frac{-1}{\sqrt{1-\dfrac{z^2}{x^2+y^2+z^2}}}\left[\frac{-xz}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{\cos\phi\cos\theta}{\Re},$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-\Re\sin\phi\sin\theta}{\Re^2\sin^2\phi} = -\frac{\sin\theta}{\Re\sin\phi},$$

and therefore

$$\frac{\partial v_x}{\partial x} = \sin\phi\cos\theta\frac{\partial v_x}{\partial \Re} + \frac{\cos\phi\cos\theta}{\Re}\frac{\partial v_x}{\partial \phi} - \frac{\sin\theta}{\Re\sin\phi}\frac{\partial v_x}{\partial \theta}.$$

Similarly,

$$\frac{\partial v_y}{\partial y} = \sin\phi\sin\theta\frac{\partial v_y}{\partial \Re} + \frac{\cos\phi\sin\theta}{\Re}\frac{\partial v_y}{\partial \phi} + \frac{\cos\theta}{\Re\sin\phi}\frac{\partial v_y}{\partial \theta},$$

$$\frac{\partial v_z}{\partial z} = \cos\phi\frac{\partial v_z}{\partial \Re} - \frac{\sin\phi}{\Re}\frac{\partial v_z}{\partial \phi},$$

When we substitute for v_x, v_y, and v_z from equations 14.73,

$$\begin{aligned}
\frac{\partial v_x}{\partial x} &= \sin\phi\cos\theta\frac{\partial}{\partial \Re}(v_\Re\sin\phi\cos\theta - v_\theta\sin\theta + v_\phi\cos\phi\cos\theta)\\
&\quad + \frac{\cos\phi\cos\theta}{\Re}\frac{\partial}{\partial \phi}(v_\Re\sin\phi\cos\theta - v_\theta\sin\theta + v_\phi\cos\phi\cos\theta)\\
&\quad - \frac{\sin\theta}{\Re\sin\phi}\frac{\partial}{\partial \theta}(v_\Re\sin\phi\cos\theta - v_\theta\sin\theta + v_\phi\cos\phi\cos\theta)\\
&= \sin\phi\cos\theta\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \Re} - \sin\theta\frac{\partial v_\theta}{\partial \Re} + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \Re}\right)\\
&\quad + \frac{\cos\phi\cos\theta}{\Re}\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \phi} + v_\Re\cos\phi\cos\theta - \sin\theta\frac{\partial v_\theta}{\partial \phi}\right.\\
&\qquad\left. + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \phi} - v_\phi\sin\phi\cos\theta\right)\\
&\quad - \frac{\sin\theta}{\Re\sin\phi}\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \theta} - v_\Re\sin\phi\sin\theta - \sin\theta\frac{\partial v_\theta}{\partial \theta}\right.\\
&\qquad\left. - v_\theta\cos\theta + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \theta} - v_\phi\cos\phi\sin\theta\right),
\end{aligned}$$

$$\frac{\partial v_y}{\partial y} = \sin\phi\sin\theta\frac{\partial}{\partial\Re}(v_{\Re}\sin\phi\sin\theta + v_\theta\cos\theta + v_\phi\cos\phi\sin\theta)$$

$$+ \frac{\cos\phi\sin\theta}{\Re}\frac{\partial}{\partial\phi}(v_{\Re}\sin\phi\sin\theta + v_\theta\cos\theta + v_\phi\cos\phi\sin\theta)$$

$$+ \frac{\cos\theta}{\Re\sin\phi}\frac{\partial}{\partial\theta}(v_{\Re}\sin\phi\sin\theta + v_\theta\cos\theta + v_\phi\cos\phi\sin\theta)$$

$$= \sin\phi\sin\theta\left(\sin\phi\sin\theta\frac{\partial v_{\Re}}{\partial\Re} + \cos\theta\frac{\partial v_\theta}{\partial\Re} + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial\Re}\right)$$

$$+ \frac{\cos\phi\sin\theta}{\Re}\left(\sin\phi\sin\theta\frac{\partial v_{\Re}}{\partial\phi} + v_{\Re}\cos\phi\sin\theta + \cos\theta\frac{\partial v_\theta}{\partial\phi}\right.$$

$$\left. + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial\phi} - v_\phi\sin\phi\sin\theta\right)$$

$$+ \frac{\cos\theta}{\Re\sin\phi}\left(\sin\phi\sin\theta\frac{\partial v_{\Re}}{\partial\theta} + v_{\Re}\sin\phi\cos\theta + \cos\theta\frac{\partial v_\theta}{\partial\theta}\right.$$

$$\left. - v_\theta\sin\theta + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial\theta} + v_\phi\cos\phi\cos\theta\right),$$

$$\frac{\partial v_z}{\partial z} = \cos\phi\frac{\partial}{\partial\Re}(v_{\Re}\cos\phi - v_\phi\sin\phi) - \frac{\sin\phi}{\Re}\frac{\partial}{\partial\phi}(v_{\Re}\cos\phi - v_\phi\sin\phi)$$

$$= \cos\phi\left(\cos\phi\frac{\partial v_{\Re}}{\partial\Re} - \sin\phi\frac{\partial v_\phi}{\partial\Re}\right) - \frac{\sin\phi}{\Re}\left(\cos\phi\frac{\partial v_{\Re}}{\partial\phi} - v_{\Re}\sin\phi - \sin\phi\frac{\partial v_\phi}{\partial\phi} - v_\phi\cos\phi\right).$$

When these are added together and simplified, the result is

$$\text{div}\,\mathbf{v} = \frac{\partial v_{\Re}}{\partial\Re} + \frac{2}{\Re}v_{\Re} + \frac{1}{\Re}\frac{\partial v_\phi}{\partial\phi} + \frac{\cot\phi}{\Re}v_\phi + \frac{1}{\Re\sin\phi}\frac{\partial v_\theta}{\partial\theta}. \tag{14.79a}$$

This can be expressed in a form resembling that for divergence in Cartesian coordinates,

$$\text{div}\,\mathbf{v} = \frac{1}{\Re^2\sin\phi}\left[\frac{\partial}{\partial\Re}(\Re^2\sin\phi\, v_{\Re}) + \frac{\partial}{\partial\phi}(\Re\sin\phi\, v_\phi) + \frac{\partial}{\partial\theta}(\Re v_\theta)\right]. \tag{14.79b}$$

Thus, to find the divergence of a vector function in spherical coordinates, we use 14.78 when its Cartesian components v_x, v_y, and v_z are known, and then substitute for x, y, and z in terms of $\Re$, ϕ, and θ, or we use 14.79 when its spherical components $v_{\Re}$, v_ϕ, and v_θ are known.

EXAMPLE 14.32

Find div **v** in spherical coordinates using 14.78 and 14.79 when $\mathbf{v} = (x^2 + y^2)\hat{\mathbf{i}} + z\hat{\mathbf{k}}$.

SOLUTION Using 14.78,

$$\text{div}\,\mathbf{v} = 2x + 1 = 2\Re\sin\phi\cos\theta + 1.$$

Alternatively, since

$$v_{\Re} = (x^2 + y^2)\sin\phi\cos\theta + z\cos\phi = \Re^2\sin^3\phi\cos\theta + \Re\cos^2\phi,$$

$$v_\phi = (x^2 + y^2)\cos\phi\cos\theta - z\sin\phi = \Re^2\sin^2\phi\cos\phi\cos\theta - \Re\sin\phi\cos\phi,$$

$$v_\theta = -(x^2 + y^2)\sin\theta = -\Re^2\sin^2\phi\sin\theta,$$

equation 14.79a gives

$$\begin{aligned}\operatorname{div}\mathbf{v} &= 2\Re\sin^3\phi\cos\theta + \cos^2\phi + \frac{2}{\Re}(\Re^2\sin^3\phi\cos\theta + \Re\cos^2\phi)\\ &\quad + \frac{1}{\Re}(2\Re^2\sin\phi\cos^2\phi\cos\theta - \Re^2\sin^3\phi\cos\theta - \Re\cos^2\phi + \Re\sin^2\phi)\\ &\quad + \frac{1}{\Re\sin\phi}(-\Re^2\sin^2\phi\cos\theta)\\ &= 2\Re\sin\phi\cos\theta + 1.\end{aligned}$$

The curl of a vector function $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{k}}. \tag{14.80}$$

It can be changed to spherical coordinates by replacing x, y, and z with $\Re$, ϕ, and θ, and transforming components according to equations 14.72. Alternatively, according to 14.72, the spherical components of $\nabla\times\mathbf{v}$ are

$$(\nabla\times\mathbf{v})_{\Re} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\sin\phi\cos\theta + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\sin\phi\sin\theta + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\cos\phi,$$

$$(\nabla\times\mathbf{v})_{\phi} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\cos\phi\cos\theta + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\cos\phi\sin\theta - \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\sin\phi,$$

$$(\nabla\times\mathbf{v})_{\theta} = -\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\sin\theta + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\cos\theta.$$

Using chain rules gives

$$\begin{aligned}\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} &= \left(\frac{\partial v_z}{\partial \Re}\frac{\partial \Re}{\partial y} + \frac{\partial v_z}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial v_z}{\partial \theta}\frac{\partial \theta}{\partial y}\right) - \left(\frac{\partial v_y}{\partial \Re}\frac{\partial \Re}{\partial z} + \frac{\partial v_y}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial v_y}{\partial \theta}\frac{\partial \theta}{\partial z}\right)\\ &= \sin\phi\sin\theta\left(\cos\phi\frac{\partial v_{\Re}}{\partial \Re} - \sin\phi\frac{\partial v_{\phi}}{\partial \Re}\right) + \frac{\cos\phi\sin\theta}{\Re}\left(\cos\phi\frac{\partial v_{\Re}}{\partial \phi} - v_{\Re}\sin\phi\right.\\ &\quad \left. - \sin\phi\frac{\partial v_{\phi}}{\partial \phi} - v_{\phi}\cos\phi\right) + \frac{\cos\theta}{\Re\sin\phi}\left(\cos\phi\frac{\partial v_{\Re}}{\partial \theta} - \sin\phi\frac{\partial v_{\phi}}{\partial \theta}\right)\\ &\quad - \cos\phi\left(\sin\phi\sin\theta\frac{\partial v_{\Re}}{\partial \Re} + \cos\phi\sin\theta\frac{\partial v_{\phi}}{\partial \Re} + \cos\theta\frac{\partial v_{\theta}}{\partial \Re}\right)\\ &\quad + \frac{\sin\phi}{\Re}\left(\sin\phi\sin\theta\frac{\partial v_{\Re}}{\partial \phi} + v_{\Re}\cos\phi\sin\theta + \cos\phi\sin\theta\frac{\partial v_{\phi}}{\partial \phi}\right.\\ &\quad \left. - v_{\phi}\sin\phi\sin\theta + \cos\theta\frac{\partial v_{\theta}}{\partial \phi}\right);\end{aligned}$$

$$\begin{aligned}
\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} &= \left(\frac{\partial v_x}{\partial \Re}\frac{\partial \Re}{\partial z} + \frac{\partial v_x}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial v_x}{\partial \theta}\frac{\partial \theta}{\partial z}\right) - \left(\frac{\partial v_z}{\partial \Re}\frac{\partial \Re}{\partial x} + \frac{\partial v_z}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial v_z}{\partial \theta}\frac{\partial \theta}{\partial x}\right) \\
&= \cos\phi\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \Re} - \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \Re} - \sin\theta\frac{\partial v_\theta}{\partial \Re}\right) - \frac{\sin\phi}{\Re}\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \phi}\right. \\
&\qquad \left. + v_\Re\cos\phi\cos\theta + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \phi} - v_\phi\sin\phi\cos\theta - \sin\theta\frac{\partial v_\theta}{\partial \phi}\right) \\
&\quad - \sin\phi\cos\theta\left(\cos\phi\frac{\partial v_\Re}{\partial \Re} - \sin\phi\frac{\partial v_\phi}{\partial \Re}\right) \\
&\quad - \frac{\cos\phi\cos\theta}{\Re}\left(\cos\phi\frac{\partial v_\Re}{\partial \phi} - v_\Re\sin\phi - \sin\phi\frac{\partial v_\phi}{\partial \phi} - v_\phi\cos\phi\right) \\
&\quad + \frac{\sin\theta}{\Re\sin\phi}\left(\cos\phi\frac{\partial v_\Re}{\partial \theta} - \sin\phi\frac{\partial v_\phi}{\partial \theta}\right);
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} &= \left(\frac{\partial v_y}{\partial \Re}\frac{\partial \Re}{\partial x} + \frac{\partial v_y}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial v_y}{\partial \theta}\frac{\partial \theta}{\partial x}\right) - \left(\frac{\partial v_x}{\partial \Re}\frac{\partial \Re}{\partial y} + \frac{\partial v_x}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial v_x}{\partial \theta}\frac{\partial \theta}{\partial y}\right) \\
&= \sin\phi\cos\theta\left(\sin\phi\sin\theta\frac{\partial v_\Re}{\partial \Re} + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial \Re} + \cos\theta\frac{\partial v_\theta}{\partial \Re}\right) \\
&\quad + \frac{\cos\phi\cos\theta}{\Re}\left(\sin\phi\sin\theta\frac{\partial v_\Re}{\partial \phi} + v_\Re\cos\phi\sin\theta + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial \phi}\right. \\
&\quad \left. - v_\phi\sin\phi\sin\theta + \cos\theta\frac{\partial v_\theta}{\partial \phi}\right) - \frac{\sin\theta}{\Re\sin\phi}\left(\sin\phi\sin\theta\frac{\partial v_\Re}{\partial \theta} + v_\Re\sin\phi\cos\theta\right. \\
&\quad \left. + \cos\phi\sin\theta\frac{\partial v_\phi}{\partial \theta} + v_\phi\cos\phi\cos\theta + \cos\theta\frac{\partial v_\theta}{\partial \theta} - v_\theta\sin\theta\right) \\
&\quad - \sin\phi\sin\theta\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \Re} + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \Re} - \sin\theta\frac{\partial v_\theta}{\partial \Re}\right) \\
&\quad - \frac{\cos\phi\sin\theta}{\Re}\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \phi} + v_\Re\cos\phi\cos\theta\right. \\
&\quad \left. + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \phi} - v_\phi\sin\phi\cos\theta - \sin\theta\frac{\partial v_\theta}{\partial \phi}\right) \\
&\quad - \frac{\cos\theta}{\Re\sin\phi}\left(\sin\phi\cos\theta\frac{\partial v_\Re}{\partial \theta} - v_\Re\sin\phi\sin\theta + \cos\phi\cos\theta\frac{\partial v_\phi}{\partial \theta}\right. \\
&\quad \left. - v_\phi\cos\phi\sin\theta - \sin\theta\frac{\partial v_\theta}{\partial \theta} - v_\theta\cos\theta\right).
\end{aligned}$$

These can be simplified to

$$\begin{aligned}
(\nabla \times \mathbf{v})_\Re &= \frac{1}{\Re}\frac{\partial v_\theta}{\partial \phi} - \frac{1}{\Re\sin\phi}\frac{\partial v_\phi}{\partial \theta} + \frac{\cot\phi}{\Re}v_\theta, \\
(\nabla \times \mathbf{v})_\phi &= \frac{1}{\Re\sin\phi}\frac{\partial v_\Re}{\partial \theta} - \frac{\partial v_\theta}{\partial \Re} - \frac{1}{\Re}v_\theta,
\end{aligned}$$

$$(\nabla \times \mathbf{v})_\theta = -\frac{1}{\Re}\frac{\partial v_\Re}{\partial \phi} + \frac{\partial v_\phi}{\partial \Re} + \frac{1}{\Re}v_\phi.$$

Thus,

$$\text{curl}\,\mathbf{v} = \frac{1}{\Re}\left(\frac{\partial v_\theta}{\partial \phi} - \frac{1}{\sin\phi}\frac{\partial v_\phi}{\partial \theta} + v_\theta \cot\phi\right)\hat{\boldsymbol{\Re}} + \left(\frac{1}{\Re\sin\phi}\frac{\partial v_\Re}{\partial \theta} - \frac{\partial v_\theta}{\partial \Re} - \frac{1}{\Re}v_\theta\right)\hat{\boldsymbol{\phi}}$$
$$+ \left(\frac{\partial v_\phi}{\partial \Re} - \frac{1}{\Re}\frac{\partial v_\Re}{\partial \phi} + \frac{1}{\Re}v_\phi\right)\hat{\boldsymbol{\theta}}. \qquad (14.81a)$$

This can be expressed in determinant form as is the case in Cartesian coordinates. Expanding the following determinant gives

$$\frac{1}{\Re^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\Re}} & \Re\hat{\boldsymbol{\phi}} & \Re\sin\phi\hat{\boldsymbol{\theta}} \\ \dfrac{\partial}{\partial \Re} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ v_\Re & \Re v_\phi & \Re\sin\phi\, v_\theta \end{vmatrix} = \frac{1}{\Re^2\sin\phi}\left[\left(\Re\sin\phi\frac{\partial v_\theta}{\partial \phi} + \Re\cos\phi\, v_\theta - \Re\frac{\partial v_\phi}{\partial \theta}\right)\hat{\boldsymbol{\Re}}\right.$$
$$\left. + \left(\frac{\partial v_\Re}{\partial \theta} - \Re\sin\phi\frac{\partial v_\theta}{\partial \Re} - \sin\phi\, v_\theta\right)\Re\hat{\boldsymbol{\phi}}\right]$$
$$+ \left(\Re\frac{\partial v_\phi}{\partial \Re} + v_\phi - \frac{\partial v_\Re}{\partial \phi}\right)\Re\sin\phi\,\hat{\boldsymbol{\theta}}$$
$$= \frac{1}{\Re}\left(\frac{\partial v_\theta}{\partial \phi} - \frac{1}{\sin\phi}\frac{\partial v_\phi}{\partial \theta} + \cot\phi\, v_\theta\right)\hat{\boldsymbol{\Re}}$$
$$+ \left(\frac{1}{\Re\sin\phi}\frac{\partial v_\Re}{\partial \theta} - \frac{\partial v_\theta}{\partial \Re} - \frac{1}{\Re}v_\theta\right)\hat{\boldsymbol{\phi}}$$
$$+ \left(\frac{\partial v_\phi}{\partial \Re} - \frac{1}{\Re}\frac{\partial v_\Re}{\partial \phi} + \frac{1}{\Re}v_\phi\right)\hat{\boldsymbol{\theta}}.$$

Hence, we may write that

$$\text{curl}\,\mathbf{v} = \frac{1}{\Re^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\Re}} & \Re\hat{\boldsymbol{\phi}} & \Re\sin\phi\hat{\boldsymbol{\theta}} \\ \dfrac{\partial}{\partial \Re} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial \theta} \\ v_\Re & \Re v_\phi & \Re\sin\phi\, v_\theta \end{vmatrix}. \qquad (14.81b)$$

EXAMPLE 14.33

Find the spherical components of the curl of the vector function in Example 14.32 using 14.80 and 14.81b.

SOLUTION Using 14.80,

$$\text{curl}\,\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & 0 & z \end{vmatrix} = -2y\hat{\mathbf{k}}.$$

Equations 14.72 give the spherical components of this vector,

$$v_\Re = -2y\cos\phi = -2\Re\sin\phi\cos\phi\sin\theta,$$
$$v_\phi = 2y\sin\phi = 2\Re\sin^2\phi\sin\theta,$$
$$v_\theta = 0.$$

Hence,

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= -2\Re\sin\phi\cos\phi\sin\theta\,\hat{\boldsymbol{\Re}} + 2\Re\sin^2\phi\sin\theta\,\hat{\boldsymbol{\phi}}\\ &= -2\Re\sin\phi\sin\theta(\cos\phi\,\hat{\boldsymbol{\Re}} - \sin\phi\,\hat{\boldsymbol{\phi}}).\end{aligned}$$

Alternatively, using 14.81b and the spherical components of $\mathbf{v}$ calculated in Example 14.32,

$$\operatorname{curl}\mathbf{v} = \frac{1}{\Re^2\sin\phi}\begin{vmatrix}\hat{\boldsymbol{\Re}} & \Re\hat{\boldsymbol{\phi}} & \Re\sin\phi\hat{\boldsymbol{\theta}}\\ \dfrac{\partial}{\partial\Re} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial\theta}\\ \begin{matrix}\Re^2\sin^3\phi\cos\theta\\ +\Re\cos^2\phi\end{matrix} & \begin{matrix}\Re^3\sin^2\phi\cos\phi\cos\theta\\ -\Re^2\sin\phi\cos\phi\end{matrix} & -\Re^3\sin^3\phi\sin\theta\end{vmatrix}$$

$$\begin{aligned}&= \frac{1}{\Re^2\sin\phi}[(-3\Re^3\sin^2\phi\cos\phi\sin\theta + \Re^3\sin^2\phi\cos\phi\sin\theta)\hat{\boldsymbol{\Re}}\\ &\quad + (3\Re^2\sin^3\phi\sin\theta - \Re^2\sin^3\phi\sin\theta)\Re\hat{\boldsymbol{\phi}}\\ &\quad + (3\Re^2\sin^2\phi\cos\phi\cos\theta - 2\Re\sin\phi\cos\phi - 3\Re^2\sin^2\phi\cos\phi\cos\theta\\ &\qquad + 2\Re\cos\phi\sin\phi)\Re\sin\phi\,\hat{\boldsymbol{\theta}}]\\ &= 2\Re\sin\phi\sin\theta(-\cos\phi\,\hat{\boldsymbol{\Re}} + \sin\phi\,\hat{\boldsymbol{\phi}}).\end{aligned}$$

In Example 12.19 of Section 12.6 we used chain rules to find the Laplacian in polar coordinates. Chain rules could also be used to find the Laplacian in spherical coordinates. It is much easier, however, to use 14.77 and 14.79b:

$$\begin{aligned}\nabla^2 V = \nabla\cdot(\nabla V) &= \frac{1}{\Re^2\sin\phi}\left[\frac{\partial}{\partial\Re}\left(\Re^2\sin\phi\frac{\partial V}{\partial\Re}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial V}{\partial\theta}\right)\right]\\ &= \frac{1}{\Re^2}\frac{\partial}{\partial\Re}\left(\Re^2\frac{\partial V}{\partial\Re}\right) + \frac{1}{\Re^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) + \frac{1}{\Re^2\sin^2\phi}\frac{\partial^2 V}{\partial\theta^2}.\end{aligned} \quad (14.82)$$

EXERCISES 14.12

In Exercises 1–7 we develop results for cylindrical coordinates parallel to those for spherical coordinates.

1. Show that unit tangent vectors to coordinate curves in cylindrical coordinates have Cartesian components

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}, \quad \hat{\mathbf{k}} = \hat{\mathbf{k}}.$$

2. If $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{k}}$ are cylindical components of a vector with Cartesian components $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, show that

$$\begin{aligned}v_r &= v_x\cos\theta + v_y\sin\theta, & v_x &= v_r\cos\theta - v_\theta\sin\theta,\\ v_\theta &= -v_x\sin\theta + v_y\cos\theta, & v_y &= v_r\sin\theta + v_\theta\cos\theta,\\ v_z &= v_z, & v_z &= v_z.\end{aligned}$$

3. Show that the scalar and vector products of vectors $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{k}}$ and $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{k}}$ are

$$\mathbf{u}\cdot\mathbf{v} = u_r v_r + u_\theta v_\theta + u_z v_z, \qquad \mathbf{u}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{k}}\\ u_r & u_\theta & u_z\\ v_r & v_\theta & v_z\end{vmatrix}.$$

* **4.** Show that cylindrical components of the gradient of a function $f(x, y, z)$ are

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}.$$

∗ **5.** Show that the divergence of a vector function with cylindrical components $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{k}}$ is

$$\operatorname{div}\mathbf{v} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\theta}{\partial \theta} + r\frac{\partial v_z}{\partial z}\right].$$

∗ **6.** Show that the curl of a vector function with cylindrical components $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\hat{\mathbf{k}}$ is

$$\operatorname{curl}\mathbf{v} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix}.$$

∗ **7.** Use Exercises 4 and 5 to develop the Laplacian in cylindrical coordinates,

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}.$$

8. Repeat Example 14.29 for $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

9. Repeat Example 14.29 for $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and cylindrical coordinates.

10. Find the scalar and vector products of the vectors $\mathbf{u} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{v} = (x^2 + y^2)\hat{\mathbf{k}}$ in cylindrical coordinates by (a) calculating the products in Cartesian coordinates and transforming to cylindrical coordinates and (b) transforming $\mathbf{u}$ and $\mathbf{v}$ to cylindrical coordinates and using Exercise 3.

11. Repeat Example 14.30 in cylindrical coordinates.

12. Repeat Exercise 10 for spherical coordinates.

In Exercises 13–16 find cylindrical and spherical components of the vector (see Exercise 2).

13. $\mathbf{F} = 2y\hat{\mathbf{i}} - 3x\hat{\mathbf{j}} + xz\hat{\mathbf{k}}$ **14.** $\mathbf{F} = 3y\hat{\mathbf{i}} - 2z\hat{\mathbf{j}} + 2x\hat{\mathbf{k}}$

15. $\mathbf{F} = (x^2 + y^2 + z^2)\hat{\mathbf{i}} - z\hat{\mathbf{k}}$ **16.** $\mathbf{F} = x\hat{\mathbf{i}} + (x^2 + y^2)\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$

In Exercises 17–20 find components of ∇f in cylindrical and spherical coordinates (see Exercise 4 for cylindrical coordinates).

17. $f(x, y, z) = x^2 + y^2 + z^2$

∗ **18.** $f(x, y, z) = x^2 + xy + z$

∗ **19.** $f(x, y, z) = x^2 + y^2 + xyz$

∗ **20.** $f(x, y, z) = \dfrac{z}{x^2 + y^2} + e^z$

In Exercises 21–24 find the divergence of the vector function in cylindrical and spherical coordinates. Do so by (a) using 14.78 and then transforming coordinates and (b) finding cylindrical and spherical components of $\mathbf{v}$, and then using Exercise 5 and equation 14.79b.

∗ **21.** $\mathbf{v} = (x^2 + y^2)(\hat{\mathbf{i}} + \hat{\mathbf{j}})$

∗ **22.** $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

∗ **23.** $\mathbf{v} = x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$

∗ **24.** $\mathbf{v} = \sqrt{x^2 + y^2 + z^2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$

In Exercises 25–26 find the curl of the vector function in cylindrical and spherical coordinates. Do so by (a) using 14.80 and then transforming coordinates and (b) finding cylindrical and spherical components of $\mathbf{v}$, and then using Exercise 6 and equation 14.81b.

∗ **25.** $\mathbf{v} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xz\hat{\mathbf{k}}$

∗ **26.** $\mathbf{v} = (x^2 + y^2 + z^2)(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

∗ **27.** When $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$ is the fluid velocity in two-dimensional, steady-state, incompressible flow, the equation of continuity demands that

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

(see equation 14.61 with ρ constant). Show that if $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\theta}$ are polar components of $\mathbf{v}$, then the equation of continuity takes the form

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} = 0.$$

∗ **28.** The velocity and acceleration of a particle with displacement $\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ (t being time) are

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}, \quad \mathbf{a} = \ddot{\mathbf{r}} = \dot{\mathbf{v}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}},$$

where "·" indicates differentiation with respect to t. Show that velocity and acceleration in cylindrical coordinates are

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{k}}, \quad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{k}}.$$

∗ **29.** The velocity and acceleration of a particle with displacement $\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ (t being time) are

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}, \quad \mathbf{a} = \ddot{\mathbf{r}} = \dot{\mathbf{v}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}},$$

where "·" indicates differentiation with respect to t. Show that velocity and acceleration in spherical coordinates are

$$\mathbf{v} = \dot{\Re}\hat{\boldsymbol{\Re}} + \Re\dot{\phi}\hat{\boldsymbol{\phi}} + \Re\sin\phi\dot{\theta}\hat{\boldsymbol{\theta}},$$

$$\begin{aligned}\mathbf{a} = {} & (\ddot{\Re} - \Re\dot{\phi}^2 - \Re\sin^2\phi\dot{\theta}^2)\hat{\boldsymbol{\Re}} + (2\dot{\Re}\dot{\phi} + \Re\ddot{\phi} \\ & - \Re\sin\phi\cos\phi\dot{\theta}^2)\hat{\boldsymbol{\phi}} + (2\dot{\Re}\dot{\theta}\sin\phi \\ & + 2\Re\dot{\phi}\dot{\theta}\cos\phi + \Re\sin\phi\ddot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned}$$

In Exercises 30–34 we develop results for an arbitrary set of orthogonal curvilinear coordinates analogous to those for spherical and cylindrical coordinates. Suppose that (u, v, w) are curvilinear coordinates in some region R of space related to Cartesian coordinates by equations

$$\begin{aligned} x &= x(u, v, w), & & u = u(x, y, z), \\ y &= y(u, v, w), & \Longleftrightarrow \quad & v = v(x, y, z), \\ z &= z(u, v, w), & & w = w(x, y, z). \end{aligned}$$

If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, let

$$\hat{\mathbf{u}} = \frac{\partial \mathbf{r}/\partial u}{|\partial \mathbf{r}/\partial u|} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u},$$

$$\hat{\mathbf{v}} = \frac{\partial \mathbf{r}/\partial v}{|\partial \mathbf{r}/\partial v|} = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v},$$

$$\hat{\mathbf{w}} = \frac{\partial \mathbf{r}/\partial w}{|\partial \mathbf{r}/\partial w|} = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w},$$

be unit tangent vectors to the coordinate curves. These vectors are mutually perpendicular when the curvilinear coordinates are orthogonal.

* **30.** Show that curvilinear components $\mathbf{q} = q_u\hat{\mathbf{u}} + q_v\hat{\mathbf{v}} + q_w\hat{\mathbf{w}}$ of a vector function with Cartesian components $\mathbf{q} = q_x\hat{\mathbf{i}} + q_y\hat{\mathbf{j}} + q_z\hat{\mathbf{k}}$ are related by

$$q_u = \frac{1}{h_u}\left(q_x\frac{\partial x}{\partial u} + q_y\frac{\partial y}{\partial u} + q_z\frac{\partial z}{\partial u}\right),$$

$$q_v = \frac{1}{h_v}\left(q_x\frac{\partial x}{\partial v} + q_y\frac{\partial y}{\partial v} + q_z\frac{\partial z}{\partial v}\right),$$

$$q_w = \frac{1}{h_w}\left(q_x\frac{\partial x}{\partial w} + q_y\frac{\partial y}{\partial w} + q_z\frac{\partial z}{\partial w}\right),$$

$$q_x = \frac{q_u}{h_u}\frac{\partial x}{\partial u} + \frac{q_v}{h_v}\frac{\partial x}{\partial v} + \frac{q_w}{h_w}\frac{\partial x}{\partial w},$$

$$q_y = \frac{q_u}{h_u}\frac{\partial y}{\partial u} + \frac{q_v}{h_v}\frac{\partial y}{\partial v} + \frac{q_w}{h_w}\frac{\partial y}{\partial w},$$

$$q_z = \frac{q_u}{h_u}\frac{\partial z}{\partial u} + \frac{q_v}{h_v}\frac{\partial z}{\partial v} + \frac{q_w}{h_w}\frac{\partial z}{\partial w}.$$

* **31.** Show that curvilinear components of the gradient of a function $f(x, y, z)$ are

$$\text{grad}\, f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\mathbf{w}}.$$

32. The divergence of a vector function with curvilinear components $\mathbf{q} = q_u\hat{\mathbf{u}} + q_v\hat{\mathbf{v}} + q_w\hat{\mathbf{w}}$ is

$$\text{div}\,\mathbf{q} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w q_u) + \frac{\partial}{\partial v}(h_u h_w q_v) + \frac{\partial}{\partial w}(h_u h_v q_w)\right].$$

Show that this reduces to equation 14.79b in spherical coordinates.

33. The curl of a vector function with curvilinear components $\mathbf{q} = q_u\hat{\mathbf{u}} + q_v\hat{\mathbf{v}} + q_w\hat{\mathbf{w}}$ is

$$\text{curl}\,\mathbf{q} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u q_u & h_v q_v & h_w q_w \end{vmatrix}.$$

Show that this reduces to equation 14.81b in spherical coordinates.

* **34.** Use Exercises 31 and 32 to show that the Laplacian in (u, v, w) coordinates is

$$\nabla^2 V = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial V}{\partial w}\right)\right].$$