CHAPTER 9

SECTION 9.7

9.7 Translation and Rotation of Axes

In Section 1.5 we discussed horizontal and vertical translations of curves and how they affect equations of the curves. For example, when the ellipse $4x^2 + 9y^2 = 36$ in Figure 9.63a is shifted 1 unit to the left and 4 units upward, the result is the ellipse in Figure 9.63b with equation $4(x + 1)^2 + 9(y - 4)^2 = 36$.

FIGURE 9.63a Ellipse $4x^2 + 9y^2 = 36$

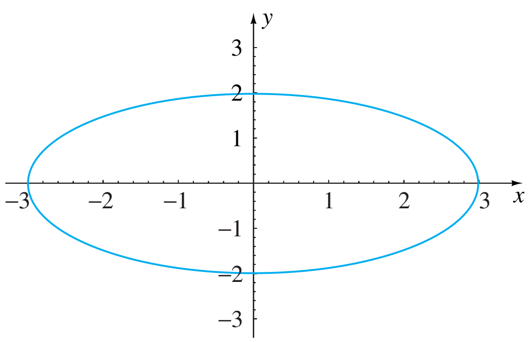
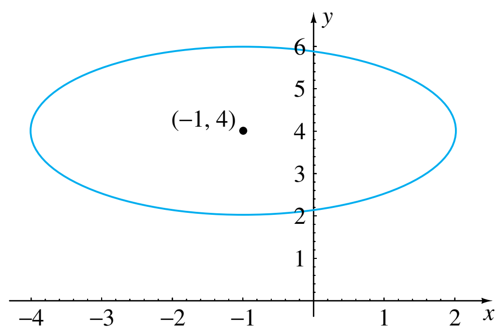


FIGURE 9.63b Translated ellipse $4(x + 1)^2 + 9(y - 4)^2 = 36$



Another way of viewing this process proves beneficial in this section. The equation $x^2 - y^2 + 4x + 10y = 5$ defines the hyperbola in Figure 9.64a. This is easily seen by completing squares on the x- and y-terms, $(y-5)^2 - (x+2)^2 = 16$. The equation is complicated because the hyperbola is not symmetric with respect to the x- and y-axes. If we choose a new coordinate system with origin $(-2, 5)$, and X- and Y-axes parallel to the x- and y-axes (Figure 9.64b), the equation of the hyperbola in this coordinate system is much simpler: $Y^2 - X^2 = 16$.

FIGURE 9.64a Hyperbola $x^2 - y^2 + 4x + 10y = 5$

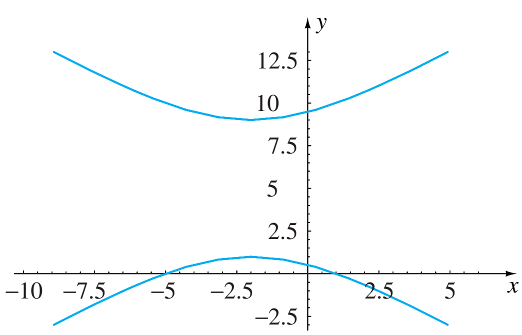


FIGURE 9.64b Using translated coordinates to simplify $x^2 - y^2 + 4x + 10y = 5$

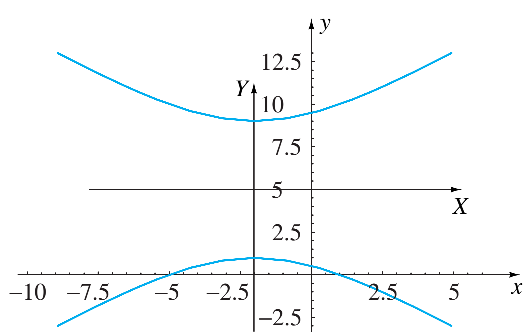
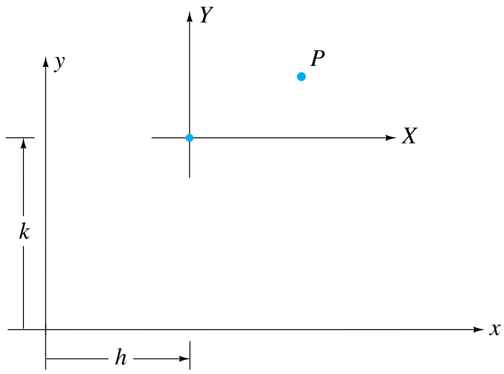


FIGURE 9.65 Relationship between translated coordinate systems



Transforming equations of curves between Cartesian coordinate systems whose **axes** are **translated** relative to one another is easily formulated in a general way. If (x, y) and (X, Y) are coordinates of point P in Figure 9.65, then

$$x = X + h, \qquad y = Y + k. \tag{9.35}$$

Given a curve with equation $F(x, y) = 0$ in the xy-coordinate system, its equation in the XY-system is $F(X + h, Y + k) = 0$. Conversely, a curve with equation $G(X, Y) = 0$ in the XY-system has equation $G(x - h, y - k) = 0$ in the xy-system.

EXAMPLE 9.28

Find a coordinate system in which the equation of the parabola $x = 2y^2 - 4y - 1$ takes the form $X = 2Y^2$.

SOLUTION The vertex of the parabola $x = 2y^2 - 4y - 1 = 2(y - 1)^2 - 3$ (Figure 9.66a) is $(-3, 1)$. If we choose X- and Y-axes through this point (Figure 9.66b), then $x = X - 3$ and $y = Y + 1$. When we substitute these into $x = 2(y - 1)^2 - 3$,

$$X - 3 = 2(Y)^2 - 3 \qquad \Longrightarrow X = 2Y^2.$$

FIGURE 9.66a Parabola $x = 2y^2 - 4y - 1$

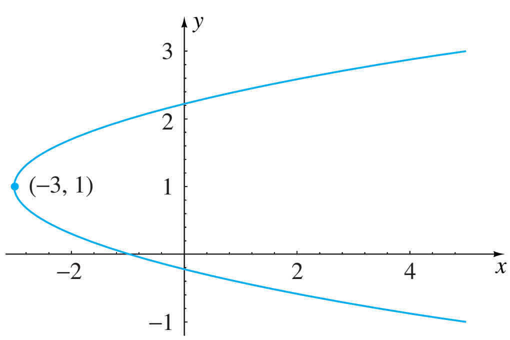
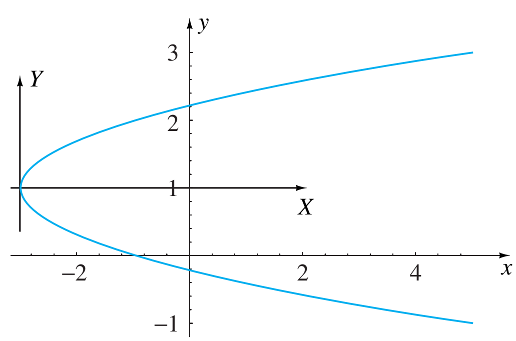


FIGURE 9.66b Using translated coordinates to simplify $x = 2y^2 - 4y - 1$



Sometimes the equation of a curve can be simplified by rotating the axes, as opposed to translating them. For example, the curve in Figure 9.67 has equation $x^2 + y^2 - 2xy - \sqrt{2}\,x - \sqrt{2}\,y + 2 = 0$; it appears to be a parabola. To be sure, we need to choose coordinate axes so that one axis is parallel to what appears to be the axis of symmetry of the curve. We discuss this in general before turning to this example.

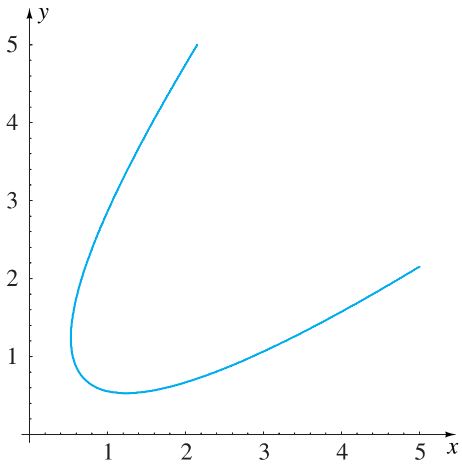


FIGURE 9.67 Parabola $x^2+y^2-2xy-\sqrt{2}x-\sqrt{2}y+2=0$

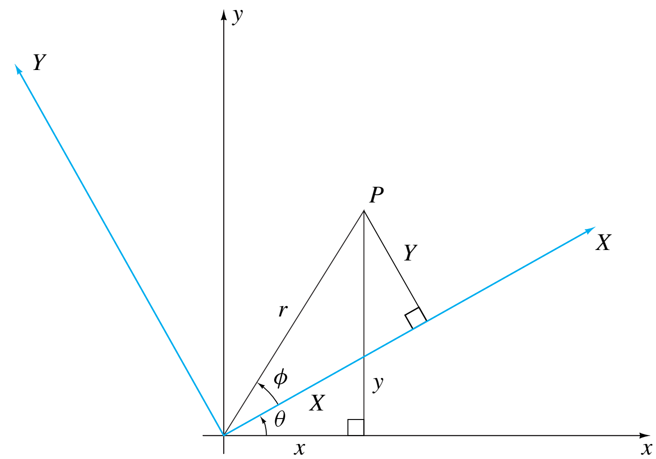


FIGURE 9.68 Relationship between rotated coordinate systems

Suppose X- and Y-axes are obtained by **rotating** x- and y-**axes** by angle θ radians (Figure 9.68). If (x, y) and (X, Y) are coordinates of a point P relative to the two sets of axes, then

$$\begin{aligned} x &= r\cos(\theta+\phi) \\ &= r(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) \\ &= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta \\ &= X\cos\theta - Y\sin\theta \end{aligned} \qquad \begin{aligned} y &= r\sin(\theta+\phi) \\ &= r(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) \\ &= (r\cos\phi)\sin\theta + (r\sin\phi)\cos\theta \\ &= X\sin\theta + Y\cos\theta. \end{aligned}$$

These equations,

$$x = X\cos\theta - Y\sin\theta, \qquad y = X\sin\theta + Y\cos\theta, \tag{9.36}$$

define (x, y)-coordinates in terms of (X, Y)-coordinates. By inverting them (or by replacing θ by $-\theta$), we get formulas for X and Y in terms of x and y,

$$X = x\cos\theta + y\sin\theta, \qquad Y = -x\sin\theta + y\cos\theta. \tag{9.37}$$

EXAMPLE 9.29

Show that the curve in Figure 9.67 is the parabola $x = y^2 + 1$ rotated $\pi/4$ radians counterclockwise around the origin.

SOLUTION To show this we find the equation of the curve with respect to the axes in Figure 9.69. According to 9.36,

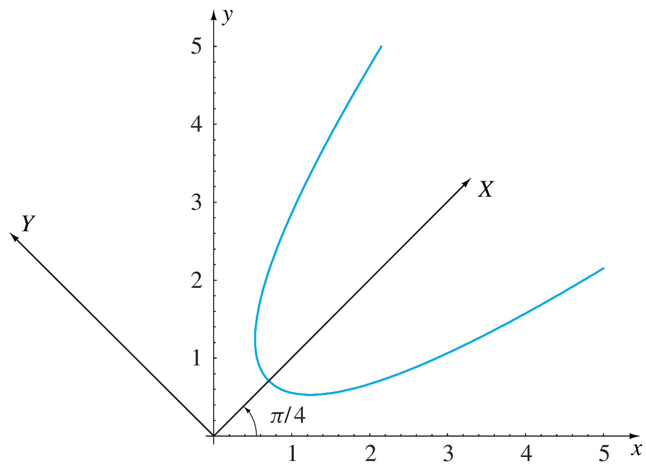
$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}, \qquad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}},$$

and when these are substituted into $x^2 + y^2 - \sqrt{2}x - \sqrt{2}y - 2xy + 2 = 0$,

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)^2 + \left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right)^2 - \sqrt{2}\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right) - \sqrt{2}\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right)$$
$$- 2\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) + 2 = 0.$$

This equation simplifies to $X = Y^2 + 1$, as required.

FIGURE 9.69 Using rotated coordinates to simplify $x^2 + y^2 - 2xy - \sqrt{2}x - \sqrt{2}y + 2 = 0$



EXAMPLE 9.30

Find the equation of the hyperbola $x^2 - y^2 = 1$ when it is rotated by $\pi/6$ radians counterclockwise around the origin.

SOLUTION Rotating the curve $\pi/6$ radians is equivalent to rotating axes by $-\pi/6$ radians (Figure 9.70a), in which case

$$x = \frac{\sqrt{3}X}{2} + \frac{Y}{2}, \qquad y = -\frac{X}{2} + \frac{\sqrt{3}Y}{2}.$$

Substitution of these into $x^2 - y^2 = 1$ gives

$$\left(\frac{\sqrt{3}X}{2} + \frac{Y}{2}\right)^2 - \left(-\frac{X}{2} + \frac{\sqrt{3}Y}{2}\right)^2 = 1,$$

which simplifies to $X^2 + 2\sqrt{3}XY - Y^2 = 2$. This is the equation of the hyperbola with respect to the XY-coordinates. It follows that the equation of the hyperbola in Figure 9.70b is $x^2 + 2\sqrt{3}\,xy - y^2 = 2$.

Equivalently, we could write the equation of the hyperbola in Figure 9.70c with respect to the XY-axes as $X^2 - Y^2 = 1$, and then use 9.37 with $\theta = \pi/6$,

$$X = \frac{\sqrt{3}\,x}{2} + \frac{y}{2}, \qquad Y = -\frac{x}{2} + \frac{\sqrt{3}\,y}{2}.$$

Substitution of these into $X^2 - Y^2 = 1$ once again leads to $x^2 + 2\sqrt{3}xy - y^2 = 2$.

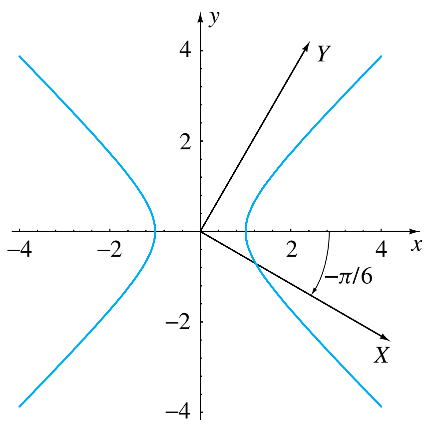


FIGURE 9.70a Instead of rotating curve counterclockwise, rotate axes clockwise

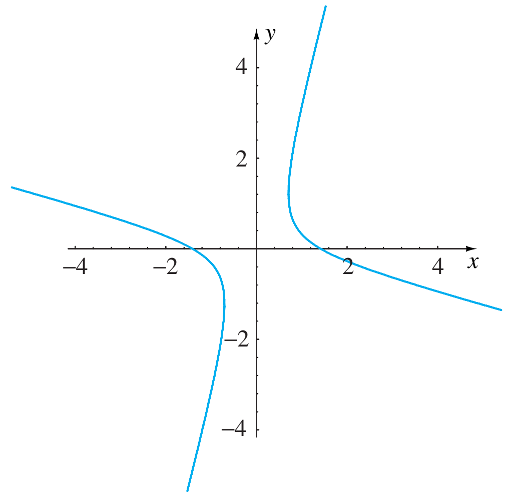


FIGURE 9.70b Results of rotating hyperbola

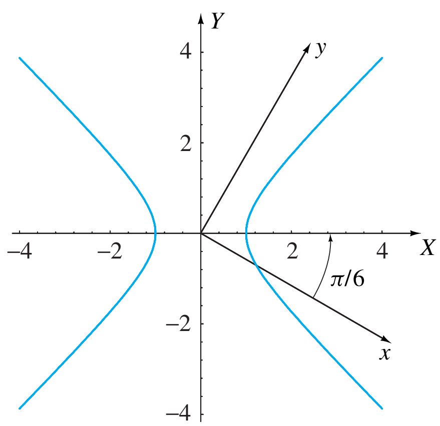


FIGURE 9.70c Rotate the curve counterclockwise

When an XY-coordinate system is translated h units in the x-direction and k units in the y-direction, and then rotated counterclockwise by θ radians (Figure 9.71), relations among x, y, X, and Y are a combination of 9.35 and 9.36:

$$x = (X + h)\cos\theta - (Y + k)\sin\theta, \qquad y = (X + h)\sin\theta + (Y + k)\cos\theta. \tag{9.38}$$

FIGURE 9.71 Relationship between rotated and translated coordinate systems

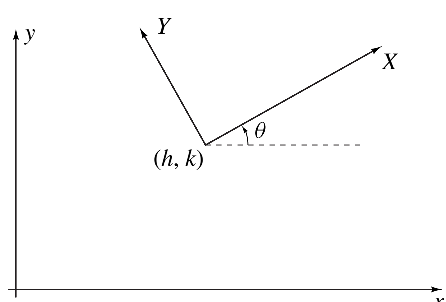


FIGURE 9.72a Ellipse in standard position

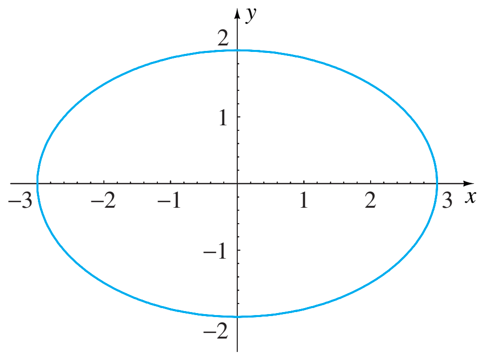


FIGURE 9.72b Translation to $(-2, 1)$ and rotated $-\pi/3$ radians

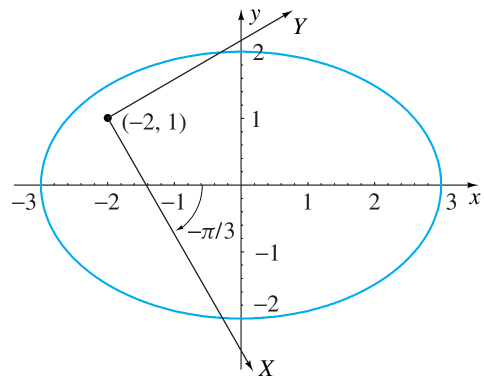
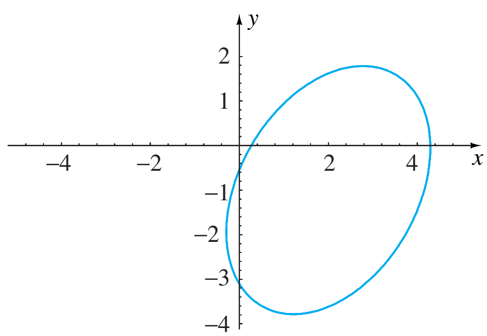


FIGURE 9.72c Translation 2 units right, 1 unit down, and rotated $\pi/3$ radians



For example, to find the equation of the ellipse $4x^2 + 9y^2 = 36$ (Figure 9.72a) when it is translated 2 units to the right, 1 unit down, and rotated clockwise $\pi/3$ radians, we draw Figure 9.72b. Coordinates are related by

$$x = \frac{1}{2}(X-2) + \frac{\sqrt{3}}{2}(Y+1), \qquad y = -\frac{\sqrt{3}}{2}(X-2) + \frac{1}{2}(Y+1).$$

Substitution of these into $4x^2 + 9y^2 = 36$ gives

$$4\left[\frac{1}{2}(X-2) + \frac{\sqrt{3}}{2}(Y+1)\right]^2 + 9\left[-\frac{\sqrt{3}}{2}(X-2) + \frac{1}{2}(Y+1)\right]^2 = 36,$$

and this simplifies to

$$31(X-2)^2 - 10\sqrt{3}(X-2)(Y+1) + 21(Y+1)^2 = 144.$$

The equation of the ellipse in Figure 9.72c is therefore

$$31(x-2)^2 - 10\sqrt{3}(x-2)(y+1) + 21(y+1)^2 = 144.$$

We have considered conic sections whose equations can be expressed in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{9.39}$$

(see Exercise 44 in Section 9.5). Parabolas, ellipses, and hyperbolas defined by 9.39 have axes of symmetry parallel to the x- and y-axes. The most general second-degree polynomial equation in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{9.40}$$

This equation also describes conic sections; the xy-term represents a rotation of the conic so that axes of symmetry are no longer parallel to the coordinate axes. To demonstrate this, we rotate axes to find a coordinate system in which the xy-term vanishes. What should remain is an equation of form 9.39.

The equation of curve 9.40 in an XY-coordinate system with axes rotated by angle θ relative to the x- and y-axes can be obtained by substituting 9.36 into 9.40:

$$\begin{aligned} A(X\cos\theta - Y\sin\theta)^2 + B(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + C(X\sin\theta + Y\cos\theta)^2 \\ + D(X\cos\theta - Y\sin\theta) + E(X\sin\theta + Y\cos\theta) + F = 0. \end{aligned}$$

This can be rearranged into the form

$$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F' = 0, \tag{9.41a}$$

where

$$\begin{aligned} A' &= A\cos^2\theta + B\sin\theta\,\cos\theta + C\sin^2\theta, \\ B' &= B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\,\cos\theta, \\ C' &= A\sin^2\theta - B\sin\theta\,\cos\theta + C\cos^2\theta, \\ D' &= D\cos\theta + E\sin\theta, \\ E' &= -D\sin\theta + E\cos\theta, \\ F' &= F. \end{aligned} \tag{9.41b}$$

To eliminate the XY-term, θ must be chosen so that

$$\begin{aligned} 0 = B' &= B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\,\cos\theta \\ &= B\cos 2\theta + (C - A)\sin 2\theta, \end{aligned}$$

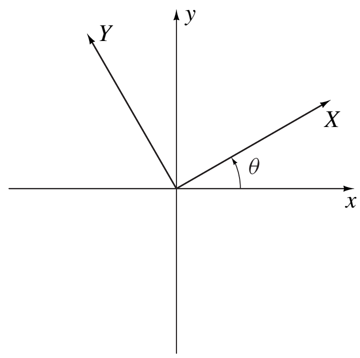
or

$$\cot 2\theta = \frac{A - C}{B}. \tag{9.42}$$

Once θ is chosen to satisfy 9.42, equation 9.41a reduces to form 9.39. To facilitate the change, in practice, we need values for $\sin\theta$ and $\cos\theta$ (for 9.41b or 9.37). We do this by restricting values of θ. In general, we can restrict θ to $-\pi < \theta < \pi$. For purposes of eliminating B', we can do more. Figures 9.73a and b show rotations by angles in the intervals $0 < \theta < \pi/2$ and $-\pi < \phi < -\pi/2$ where $\phi = \theta - \pi$. They produce exactly the same B' since $B' = B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\,\cos\theta$. This is also true for the rotations in Figure 9.73c and d, where $\phi = \theta + \pi$. What this means is that θ can be restricted to the interval $-\pi/2 < \theta < \pi/2$.

FIGURE 9.73a FIGURE 9.73b

Restriction of angle of rotation for general conic to $-\pi/2 < \theta < \pi/2$



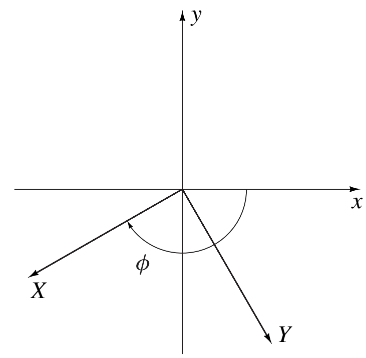
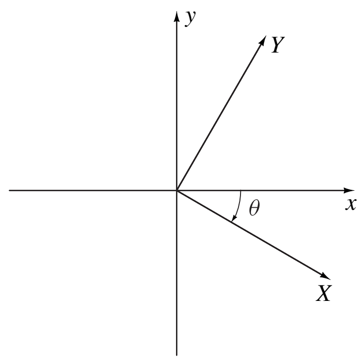
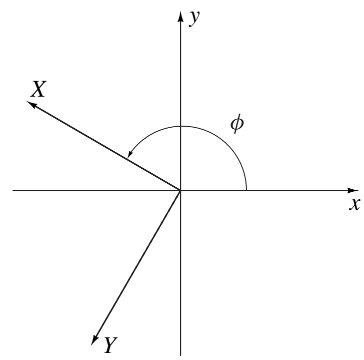


FIGURE 9.73c FIGURE 9.73d

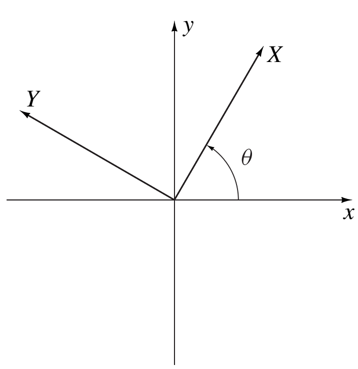


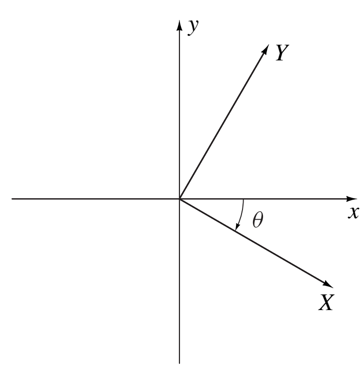


Now consider the rotations as shown in Figures 9.74, where $\phi = \theta - \pi/2$. Since $\cos(\theta - \pi/2) = \sin\theta$ and $\sin(\theta - \pi/2) = -\cos\theta$, it follows that these rotations produce B'-terms with opposite signs. That is, θ can be restricted to the interval $0 < \theta < \pi/2$ for purposes of eliminating B'.

FIGURE 9.74a FIGURE 9.74b

Restriction of angle of rotation for general conic to $0 < \theta < \pi/2$

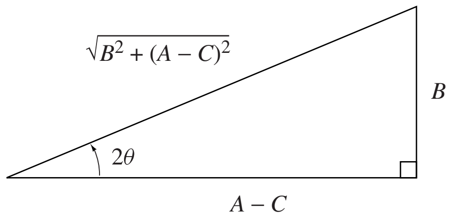




When $0 < \theta < \pi/2$, double-angle formulas 1.46b and 1.46c can be solved for

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}}, \qquad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}, \tag{9.43}$$

FIGURE 9.75 Triangle to fit $\cot 2\theta = (A - C)/B$



and these equations define exact values for $\sin\theta$ and $\cos\theta$ in terms of $\cos 2\theta$. Equation 9.42 determines $\cot 2\theta$, and therefore $\cos 2\theta$, in terms of A, B, and C. Since $\cos 2\theta$ and $\cot 2\theta$ have the same sign ($0 < 2\theta < \pi$), the triangle in Figure 9.75 leads to the formula

$$\cos 2\theta = \operatorname{sgn}\left(\frac{A - C}{B}\right)\frac{|A - C|}{\sqrt{B^2 + (A - C)^2}}, \tag{9.44a}$$

where $\operatorname{sgn}(x)$ is the signum function of Exercise 47 in Section 2.4:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Because 2θ is in the principal value range of the inverse cosine function, we can write explicitly that

$$\theta = \frac{1}{2}\operatorname{Cos}^{-1}\left[\operatorname{sgn}\left(\frac{A - C}{B}\right)\frac{|A - C|}{\sqrt{B^2 + (A - C)^2}}\right]. \tag{9.44b}$$

Let us use formulas 9.43 and 9.44 to eliminate the xy-term from some second-degree polynomial equations.

EXAMPLE 9.31

Rotate axes to eliminate the xy-term from the equation $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$. Plot the curve.

SOLUTION The angle of rotation θ that eliminates the xy-term is defined by equation 9.44b:

$$\theta = \frac{1}{2}\operatorname{Cos}^{-1}\left[\operatorname{sgn}\left(\frac{7 - 13}{-6\sqrt{3}}\right)\frac{|7 - 13|}{\sqrt{108 + (7 - 13)^2}}\right] = \frac{1}{2}\operatorname{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Axes should therefore be rotated counterclockwise $\pi/6$ radians. To find the equation of the curve with xy-term eliminated, we can use equations 9.41 or return to transformation 9.36. We illustrate both. With $A = 7$, $B = -6\sqrt{3}$, $C = 13$, and $F = -16$ in 9.41b,

$$\begin{aligned}
A' &= 7(\sqrt{3}/2)^2 - 6\sqrt{3}(1/2)(\sqrt{3}/2) + 13(1/2)^2 = 4, \\
B' &= -6\sqrt{3}(3/4 - 1/4) + 2(13 - 7)(1/2)(\sqrt{3}/2) = 0, \\
C' &= 7(1/2)^2 + 6\sqrt{3}(1/2)(\sqrt{3}/2) + 13(\sqrt{3}/2)^2 = 16, \\
D' &= 0, \\
E' &= 0, \\
F' &= -16.
\end{aligned}$$

The equation of the curve with respect to rotated coordinates X and Y is

$$4X^2 + 16Y^2 - 16 = 0 \qquad \Longrightarrow \qquad X^2 + 4Y^2 = 4.$$

Alternatively, if we substitute from 9.36, namely,

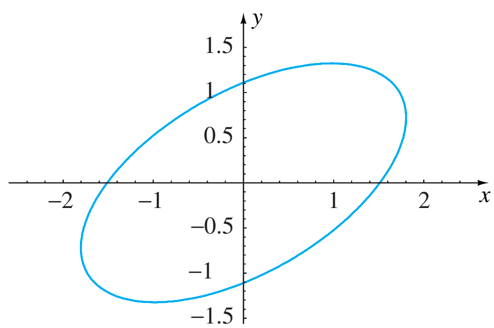
$$x = \frac{\sqrt{3}X}{2} - \frac{Y}{2}, \qquad y = \frac{X}{2} + \frac{\sqrt{3}Y}{2},$$

into $7x^2 - 6\sqrt{3}\,xy + 13y^2 = 16$,

$$7\left(\frac{\sqrt{3}X}{2} - \frac{Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}X}{2} - \frac{Y}{2}\right)\left(\frac{X}{2} + \frac{\sqrt{3}Y}{2}\right) + 13\left(\frac{X}{2} + \frac{\sqrt{3}Y}{2}\right)^2 = 16,$$

and this simplifies to $X^2 + 4Y^2 = 4$. Figure 9.76 contains a plot of the ellipse.

FIGURE 9.76 Rotation of ellipse $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$



EXAMPLE 9.32

Rotate axes to eliminate the xy-term from the equation $4x^2 - 4xy + y^2 = 5x$. Plot the curve.

SOLUTION The angle of rotation θ that eliminates the xy-term is defined by 9.44a,

$$\cos 2\theta = \operatorname{sgn}\left(\frac{4-1}{-4}\right)\frac{|4-1|}{\sqrt{16+(4-1)^2}} = -\frac{3}{5}.$$

Axes should therefore be rotated through angle θ, where according to 9.43,

$$\sin\theta = \sqrt{\frac{1-(-3/5)}{2}} = \frac{2}{\sqrt{5}}, \qquad \cos\theta = \sqrt{\frac{1+(-3/5)}{2}} = \frac{1}{\sqrt{5}}.$$

Transformation 9.36 becomes

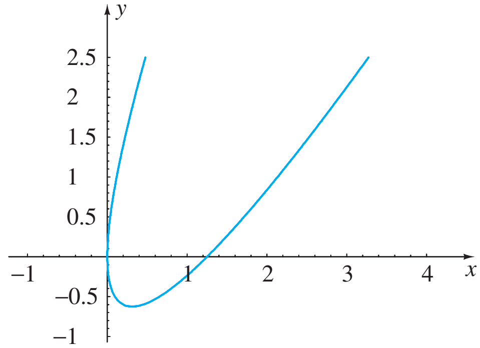
$$x = \frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}, \qquad y = \frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}},$$

and when these are substituted into $4x^2 - 4xy + y^2 = 5x$,

$$4\left(\frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}\right)\left(\frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}\right) + \left(\frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}\right)^2 = 5\left(\frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}\right).$$

This equation simplifies to $X = \sqrt{5}\,Y^2 + 2Y$, the parabola in Figure 9.77.

FIGURE 9.77 Rotation of parabola $4x^2 - 4xy + y^2 = 5x$



Classification of Conics

The xy-term in the general second-degree polynomial equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{9.45}$$

can be eliminated by rotating axes. The resulting equation then takes the form

$$A'X^2 + C'Y^2 + D'X + E'Y + F' = 0. \tag{9.46}$$

It is a straightforward calculation using 9.41b to show that, in general,

$$B'^2 - 4A'C' = B^2 - 4AC. \tag{9.47}$$

We say that $B^2 - 4AC$ is invariant under a rotation of axes. It is also invariant under translations. This fact gives us a quick way to determine whether equation 9.45 represents a parabola, an ellipse, or a hyperbola. According to Exercise 44 in Section 9.5, except for special cases, equation 9.46 is a parabola if $A'C' = 0$, an ellipse if $A'C' < 0$, and a hyperbola if $A'C' > 0$. It follows therefore that equation 9.45 should describe a parabola when $B^2 - 4AC = 0$, an ellipse when $B^2 - 4AC < 0$, and a hyperbola when $B^2 - 4AC > 0$. This is often true, but there also exist degenerate cases when 9.45 describes a pair of straight lines, a point, or nothing at all. Let us state results as follows: Except for degenerate cases, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ describes

$$\begin{aligned} &\text{an ellipse when } B^2 - 4AC < 0, \\ &\text{a parabola when } B^2 - 4AC = 0, \\ &\text{a hyperbola when } B^2 - 4AC > 0. \end{aligned} \tag{9.48}$$

Because of the possibility of degeneracy, we recommend that 9.48 not be used to determine whether 9.45 describes a parabola, an ellipse, or a hyperbola. It is preferable to transform 9.45 by means of a rotation of axes into form 9.46. We illustrate with some examples.

EXAMPLE 9.33

Determine what is described by each of the following equations:

(a) $4x^2 - 4xy + y^2 = 5x$ (b) $4x^2 - 4xy + y^2 = 5y - 3$

(c) $4x^2 - 4xy + y^2 = 5$ (d) $4x^2 - 4xy + y^2 = -2$

SOLUTION

(a) In all four cases, $B^2 - 4AC = 16 - 16 = 0$, suggesting parabolas. This is indeed the case for $4x^2 - 4xy + y^2 = 5x$ in part (a) (see Example 9.32). With the transformation

$$x = \frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}, \qquad y = \frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}$$

that rotates axes through the angle θ defined by $\cos 2\theta = -3/5$, we replace the quadratic terms $4x^2 - 4xy + y^2$ by

$$4\left(\frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X}{\sqrt{5}} - \frac{2Y}{\sqrt{5}}\right)\left(\frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}\right) + \left(\frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}\right)^2 = 5Y^2.$$

(b) With the result above, $4x^2 - 4xy + y^2 = 5y - 3$ is replaced by

$$5Y^2 = 5\left(\frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}\right) - 3 \qquad \Longrightarrow \qquad X = \frac{1}{2\sqrt{5}}(5Y^2 - \sqrt{5}\,Y + 3).$$

This is the parabola in Figure 9.78a.

(c) The equation $4x^2 - 4xy + y^2 = 5$ is transformed to $5Y^2 = 5$, or $Y = \pm 1$, the pair of straight lines in Figure 9.78b. We could also have arrived at this conclusion by writing the equation $4x^2 - 4xy + y^2 = 5$ in the form

$$(2x - y)^2 = 5 \qquad \Longrightarrow \qquad 2x - y = \pm 5.$$

These are the lines with equations $Y = \pm 1$.

(d) Equation $4x^2 - 4xy + y^2 = -2$ is replaced by $5Y^2 = -2$, an impossibility. No points satisfy $4x^2 - 4xy + y^2 = -2$. This is also obvious from the fact that $(2x - y)^2 = -2$ is an impossibility.

FIGURE 9.78a Rotation of parabola $4x^2 - 4xy + y^2 = 5y - 3$

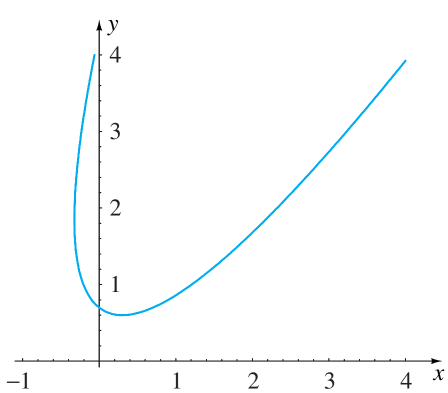
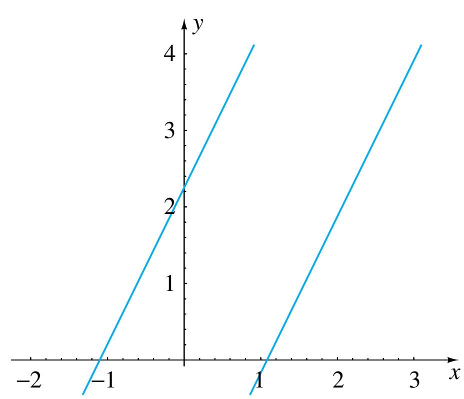


FIGURE 9.78b Straight lines $4x^2 - 4xy + y^2 = 5$



EXAMPLE 9.34

Determine what is described by the following equations:

(a) $x^2 - 3xy + y^2 + x + y = 2$ (b) $x^2 - 3xy + y^2 + x + y = 1$

SOLUTION To eliminate the xy-term, we rotate axes by the angle θ defined by

$$\cos 2\theta = \operatorname{sgn}\left(\frac{1-1}{-3}\right)\frac{|1-1|}{\sqrt{9+(1-1)^2}} = 0.$$

Thus, $2\theta = \pi/2$ or $\theta = \pi/4$. With the rotation

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}, \qquad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}},$$

$x^2 - 3xy + y^2$ is transformed to

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)^2 - 3\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) + \left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right)^2 = \frac{1}{2}(4Y^2 - X^2).$$

(a) Equation $x^2 - 3xy + y^2 + x + y = 2$ is transformed to

$$\frac{1}{2}(4Y^2 - X^2) + \left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right) + \left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 2 \implies 4Y^2 - X^2 + 2\sqrt{2}X = 4.$$

When the square is completed on the X-term, the hyperbola

$$2Y^2 - \frac{(X-\sqrt{2})^2}{2} = 1$$

in Figure 9.79a results.

(b) The equation $x^2 - 3xy + y^2 + x + y = 1$ reduces to $2Y^2 - (X - \sqrt{2})^2 = 0$. This describes the two straight lines $Y = \pm(X - \sqrt{2})/\sqrt{2}$ in Figure 9.79b.

FIGURE 9.79a Hyperbola $x^2 - 3xy + y^2 + x + y = 2$

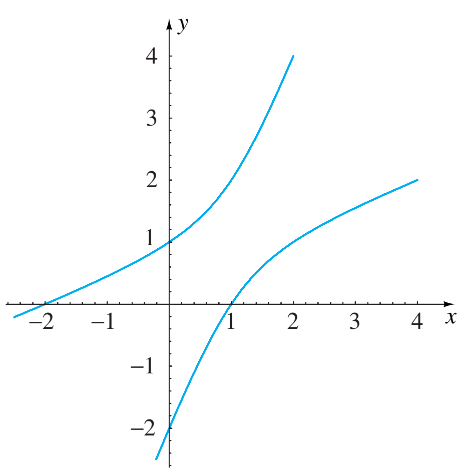
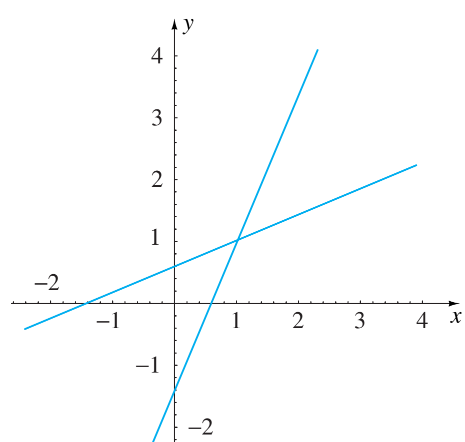


FIGURE 9.79b Straight lines $x^2 - 3xy + y^2 + x + y = 1$



EXERCISES 9.7

In Exercises 1–18 eliminate the xy-term from the equation by a suitable rotation of axes. Identify, and plot or draw, whatever is described by the equation.

1. $x^2 - 6xy + 9y^2 + x = 4$

2. $x^2 - 6xy + 9y^2 = 4$

3. $x^2 - 6xy + 9y^2 = 0$

4. $x^2 - 6xy + 9y^2 = -3$

5. $x^2 + 3xy + 5y^2 = 3$

6. $x^2 + 3xy + 5y^2 + x = 4$

7. $x^2 + 3xy + 5y^2 = 0$

8. $x^2 + 3xy + 5y^2 = -10$

9. $x^2 + 4xy + y^2 = 1$

10. $x^2 + 4xy + y^2 = x - y - 1$

11. $x^2 + 4xy + y^2 = x - y + 1/2$

12. $x^2 + 4xy + y^2 = 0$

13. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$

14. $x^2 + 6xy + 9y^2 = 16$

15. $2x^2 - 2\sqrt{2}xy + 3y^2 = 4$

16. $x^2 - 10xy + y^2 + x + y + 1 = 0$

17. $2x^2 - 4xy + 5y^2 + x - y = 4$

18. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

19. Draw the curve $xy - 2y - 4x = 0$ by (a) solving for y in terms of x and (b) rotating axes to eliminate the xy-term.

20. By factoring $x^2 + 6xy + 9y^2$, show that the equation in Exercise 14 describes two straight lines.

* **21.** Show by factoring that $6x^2 - 5xy + y^2 + y - 3x = 0$ describes two straight lines.

* **22.** Draw the curve $y^2 = 4 + \sqrt{3}xy$ by (a) solving for y in terms of x and (b) rotating axes to eliminate the xy-term.

* **23.** Paths of cellestial objects can be described by equations of the form $r = a/(1 + \epsilon\ \cos\theta)$, where $a > 0$ is a constant. To find the time to travel from one point on teh path to another, it is necesary to evaluate the integral

$$I = \int \frac{1}{1 + \epsilon\ \cos\theta)^2}\, d\theta.$$

Do so in the cases that (a) $\epsilon = 0$, (b) $0 < \epsilon < 1$, (c) $\epsilon = 1$, and (d) $\epsilon > 1$.